Polynomials

Roots and Discriminants

Find Root

$$Ax^2 + 2Bx + C = 0$$

$$Ax^3 + 3Bx^2 + 3Cx + D = 0$$

$$Ax^4 + 4Bx^3 + 6Cx^2 + 4Dx + E = 0$$

Step 1) Translate Parameter

$$x = \hat{x} - B/A$$

Find Root

$$A\hat{x}^2 + \hat{C} = 0$$

$$A\hat{x}^3 + 3\hat{C}\hat{x} + \hat{D} = 0$$

$$A\hat{x}^4 + 6\hat{C}\hat{x}^2 + 4\hat{D}\hat{x} + \hat{E} = 0$$

Sum of Roots = 0

Step 2) Solve simpler Polynomials

Step 3) Transform Back

$$x = \hat{x} - B/A$$

Homogeneous Polynomials

$$Ax^2 + 2Bxw + Cw^2 = 0$$

$$Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3 = 0$$

$$Ax^4 + 4Bx^3 + 6Cx^2w + 4Dxw^2 + Ew^3 = 0$$

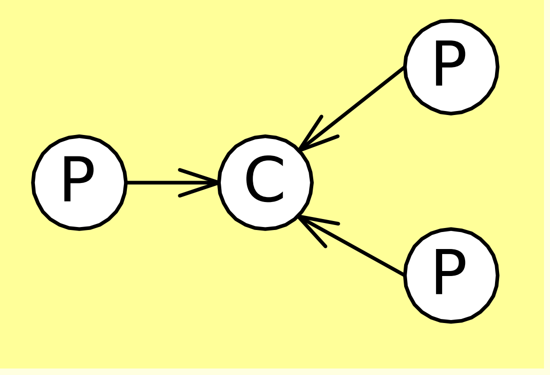
$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} \hat{x} & \hat{w} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} \hat{x} & \hat{w} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{B}{A} & 1 \end{bmatrix}$$

Solving Homogeneous Cubic Polynomials

$$Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3 = 0$$

$$\begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} & \begin{bmatrix} B & C \\ C & D \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = 0$$

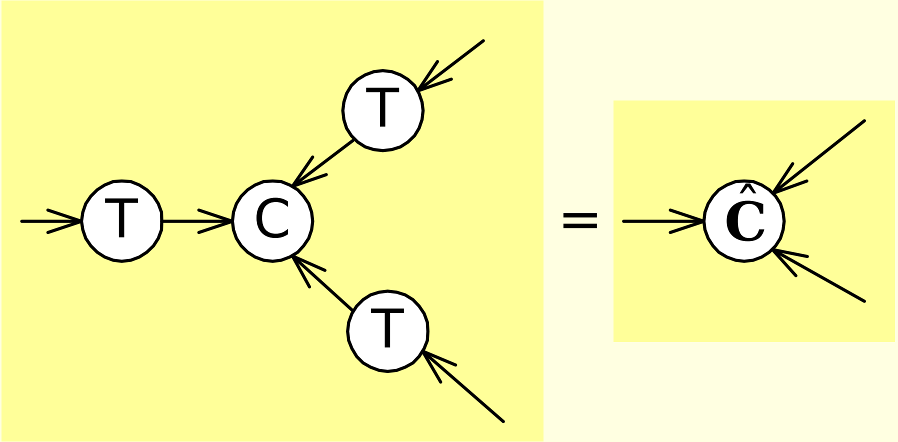


$= 0$

General Homogeneous Parameter Transform

$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} \hat{x} & \hat{w} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

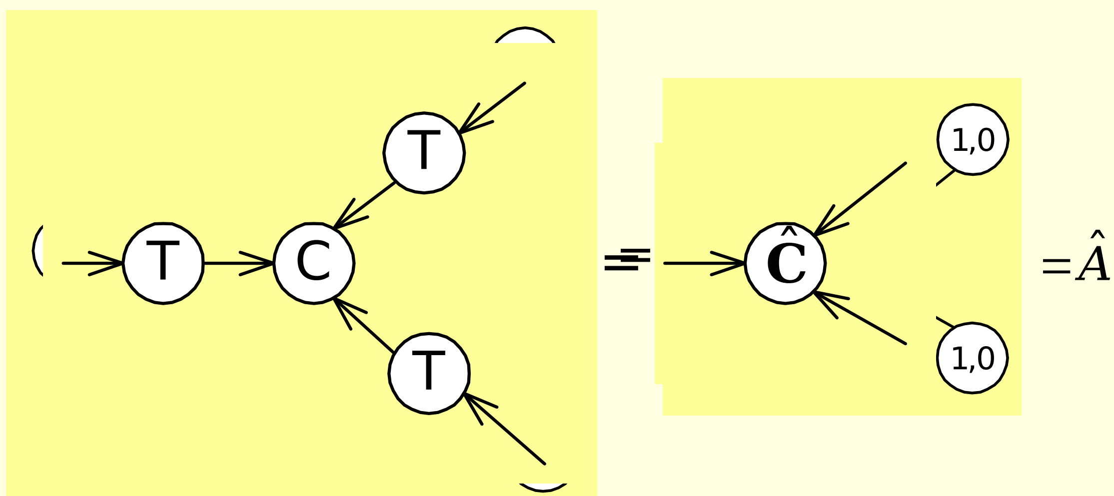
$$\mathbf{p} = \hat{\mathbf{p}}\mathbf{T}$$



Elements of Transformed C

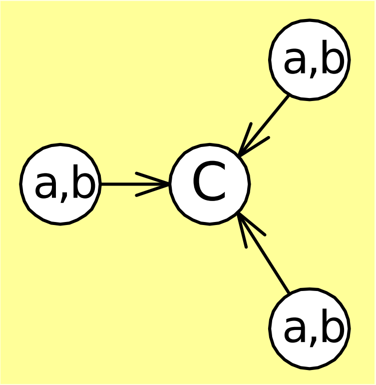
$$\mathbf{T} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} & \begin{bmatrix} B & C \\ C & D \end{bmatrix} \end{bmatrix} \to \begin{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{B} & \hat{C} \end{bmatrix} & \begin{bmatrix} \hat{B} & \hat{C} \\ \hat{C} & \hat{D} \end{bmatrix} \end{bmatrix}$$

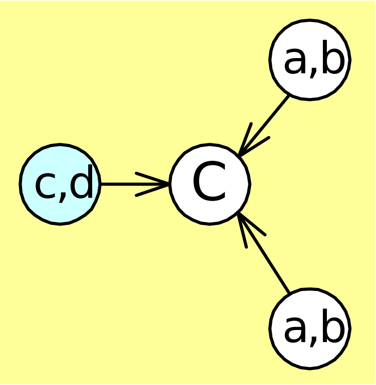


Elements of Transformed Cubic

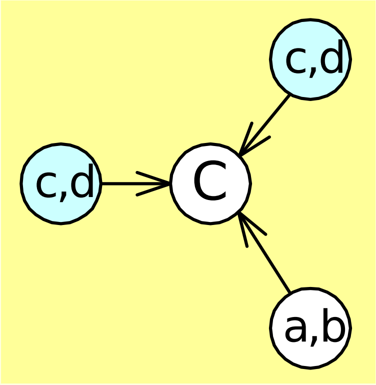
$\hat{A} =$



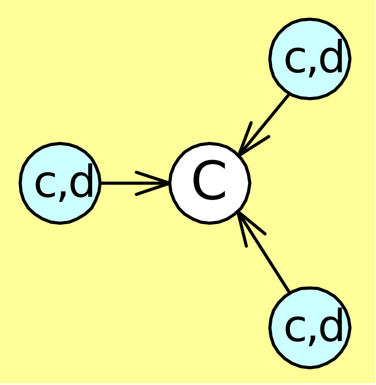
$\hat{B} =$



$\hat{C} =$

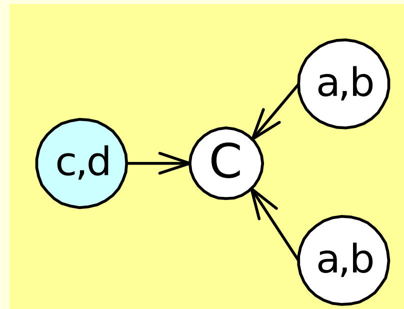


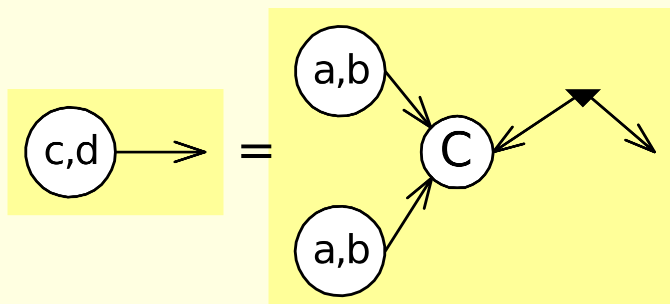
$\hat{D} =$



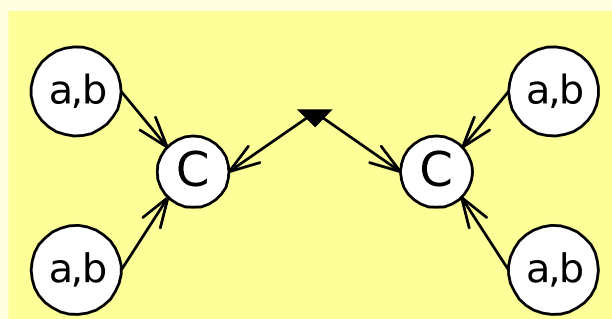
Make B^ zero

$\hat{B} =$

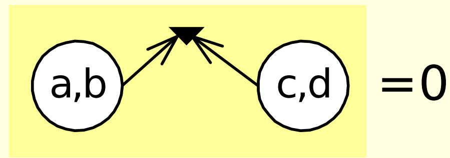


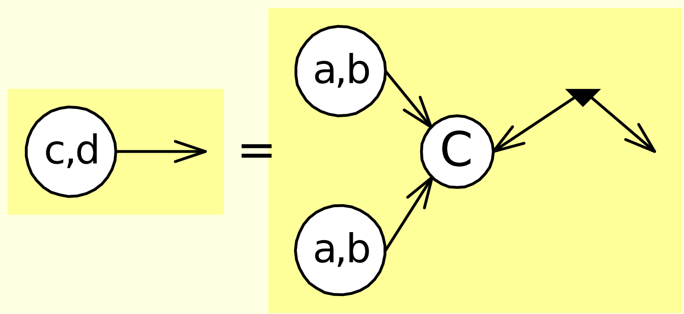


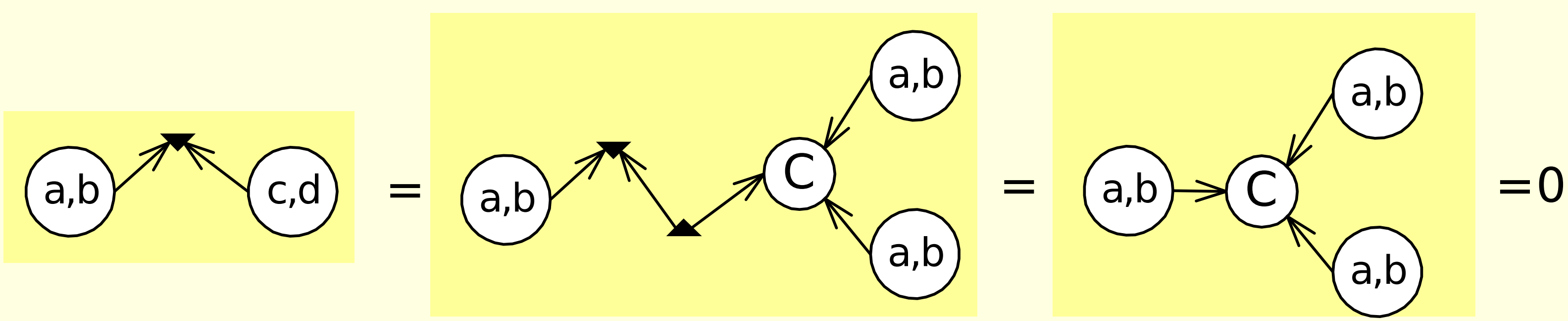
$\hat{B} =$



When is Transform Singular

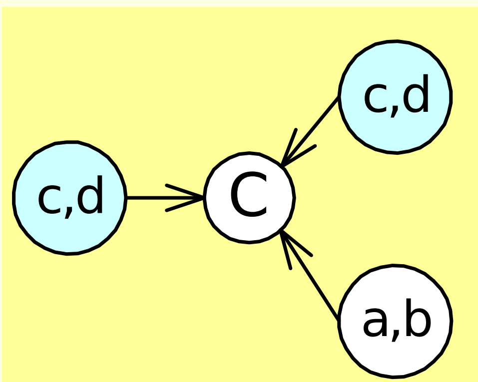






Evaluate C^

$\hat{C} =$



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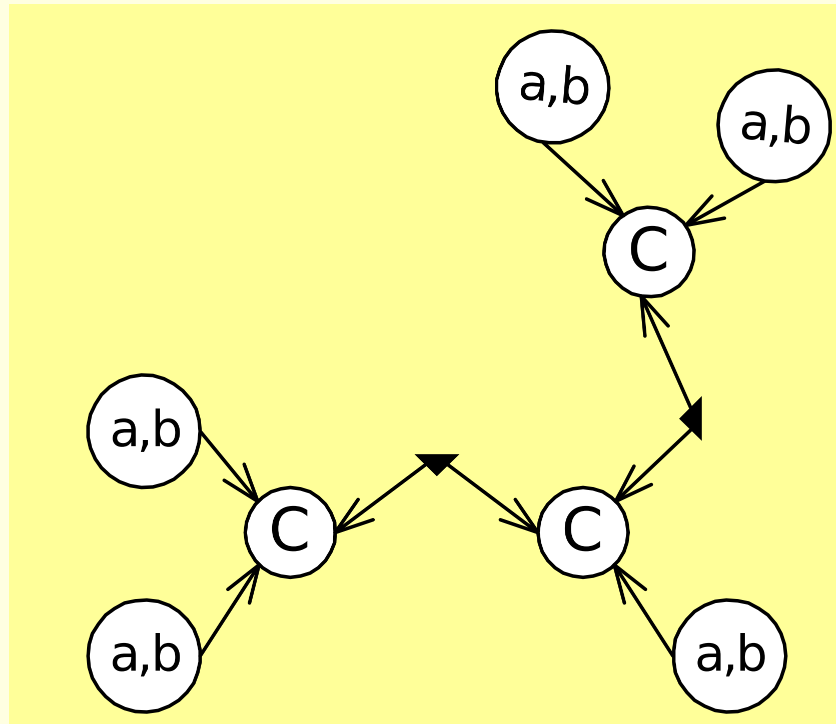
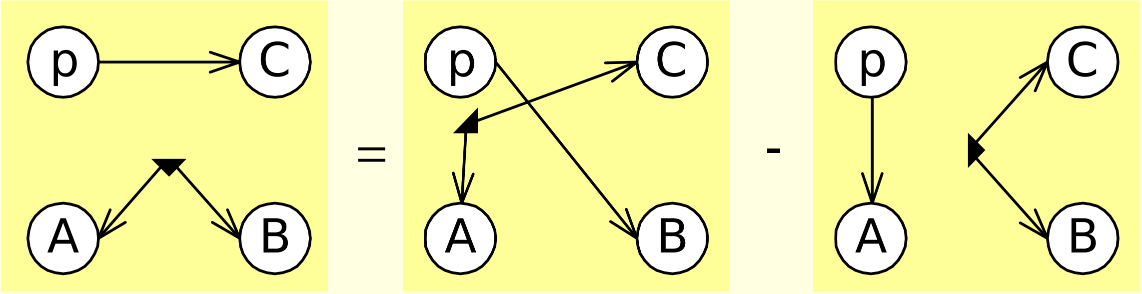
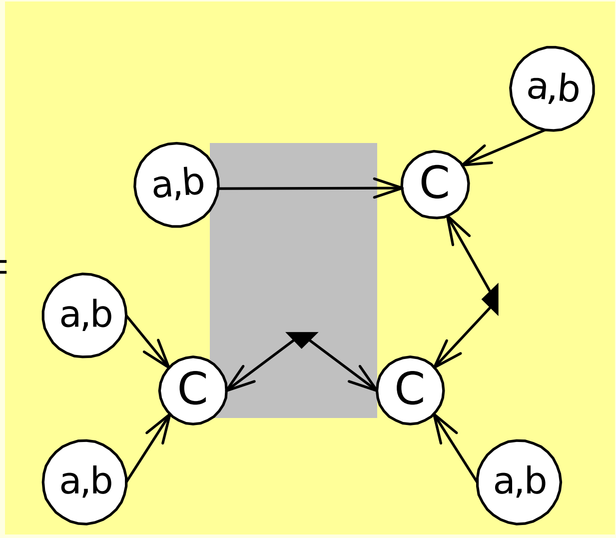
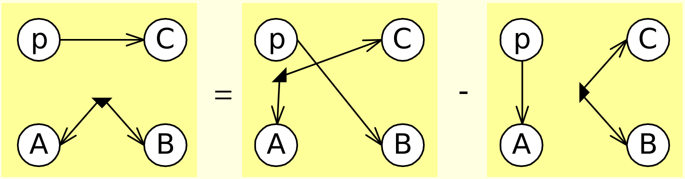


Apply variant of epsilon-delta

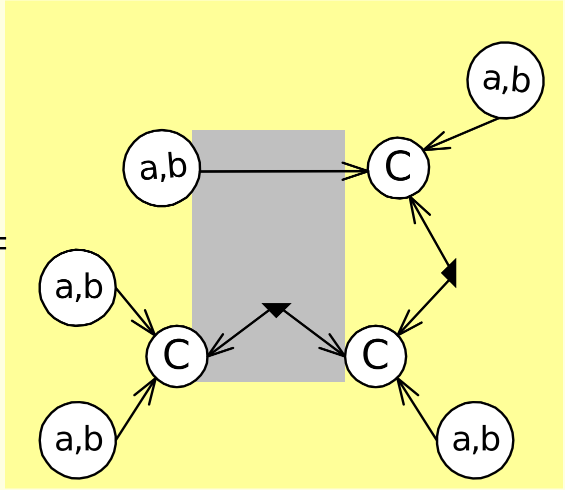


$\hat{C} =$ 

Apply



$\hat{C} =$



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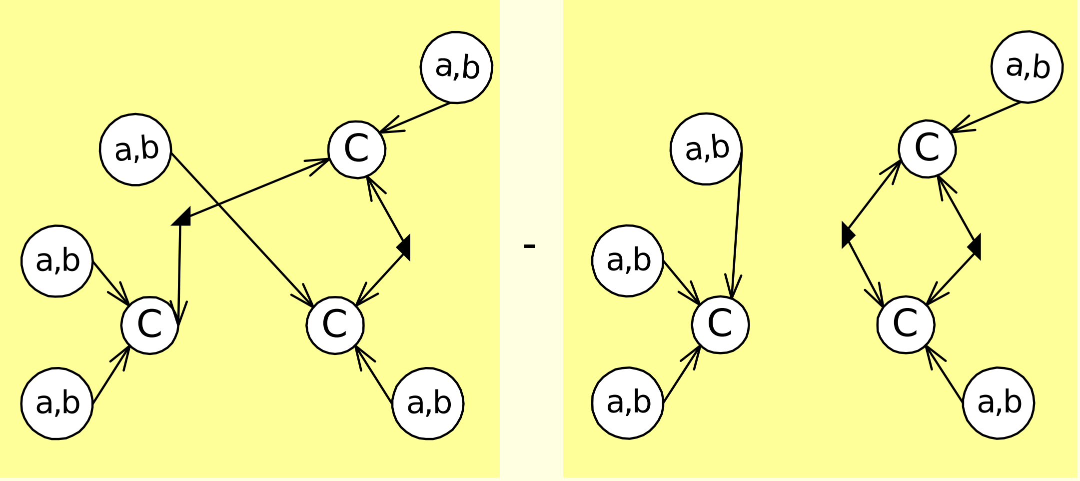
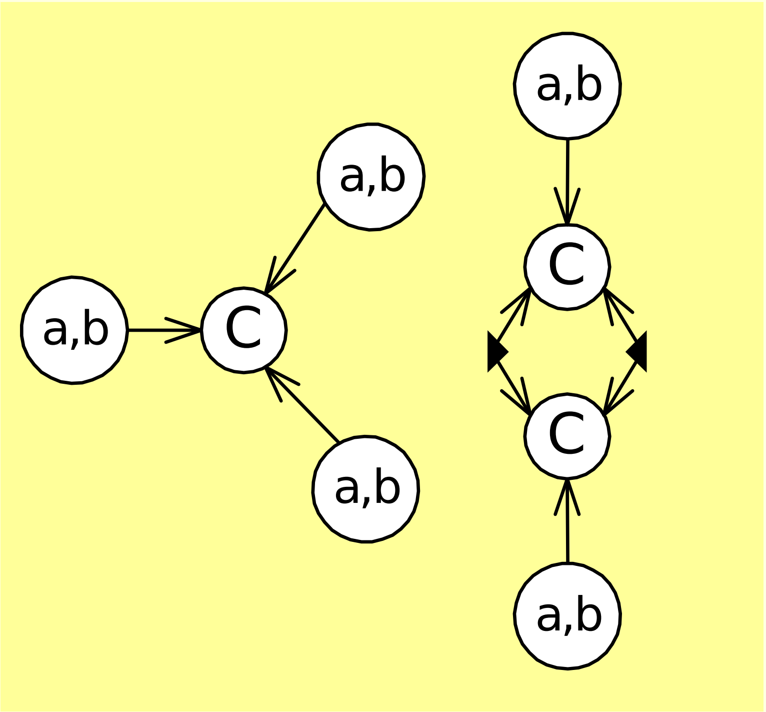


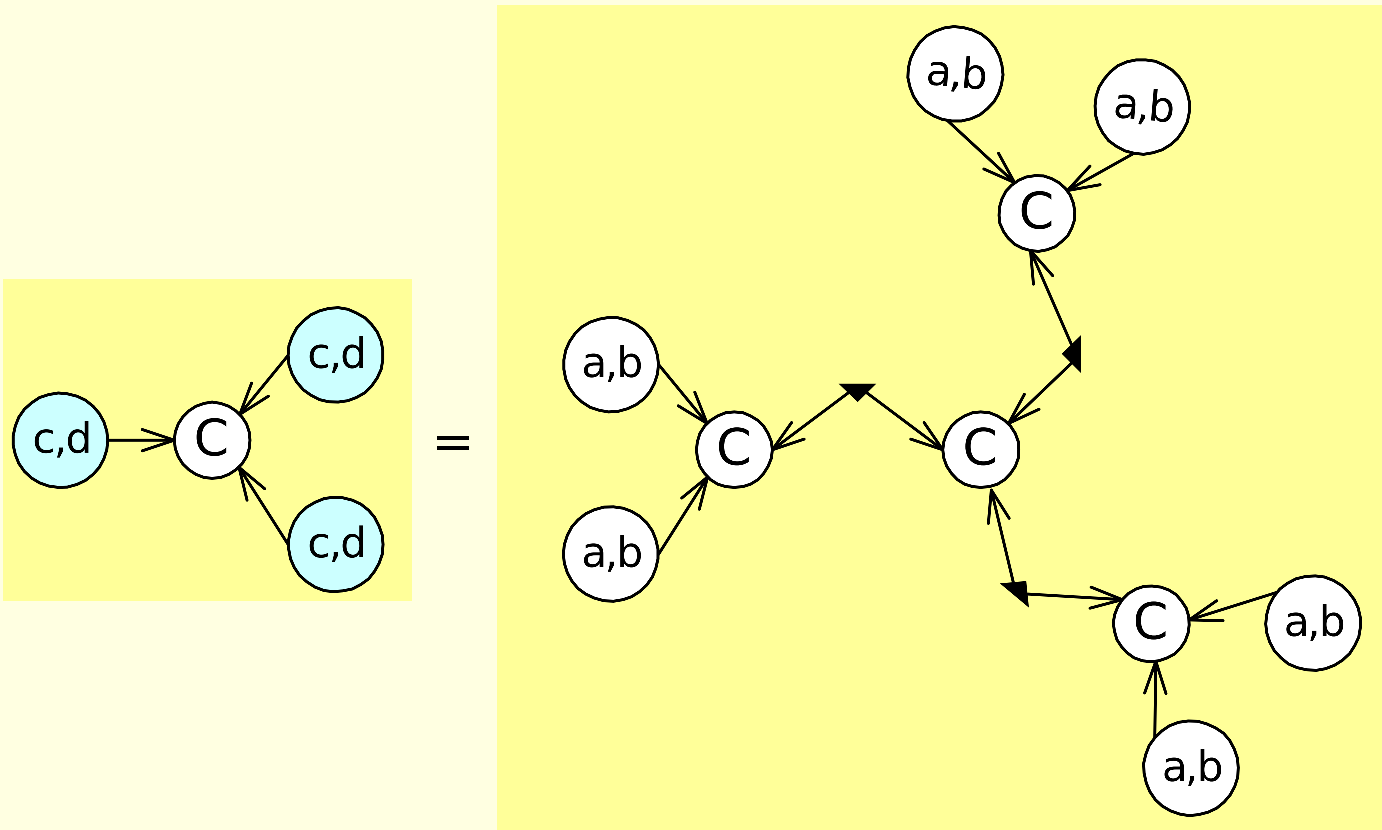
Final C^

$$\hat{C} = -\frac{1}{2}$$



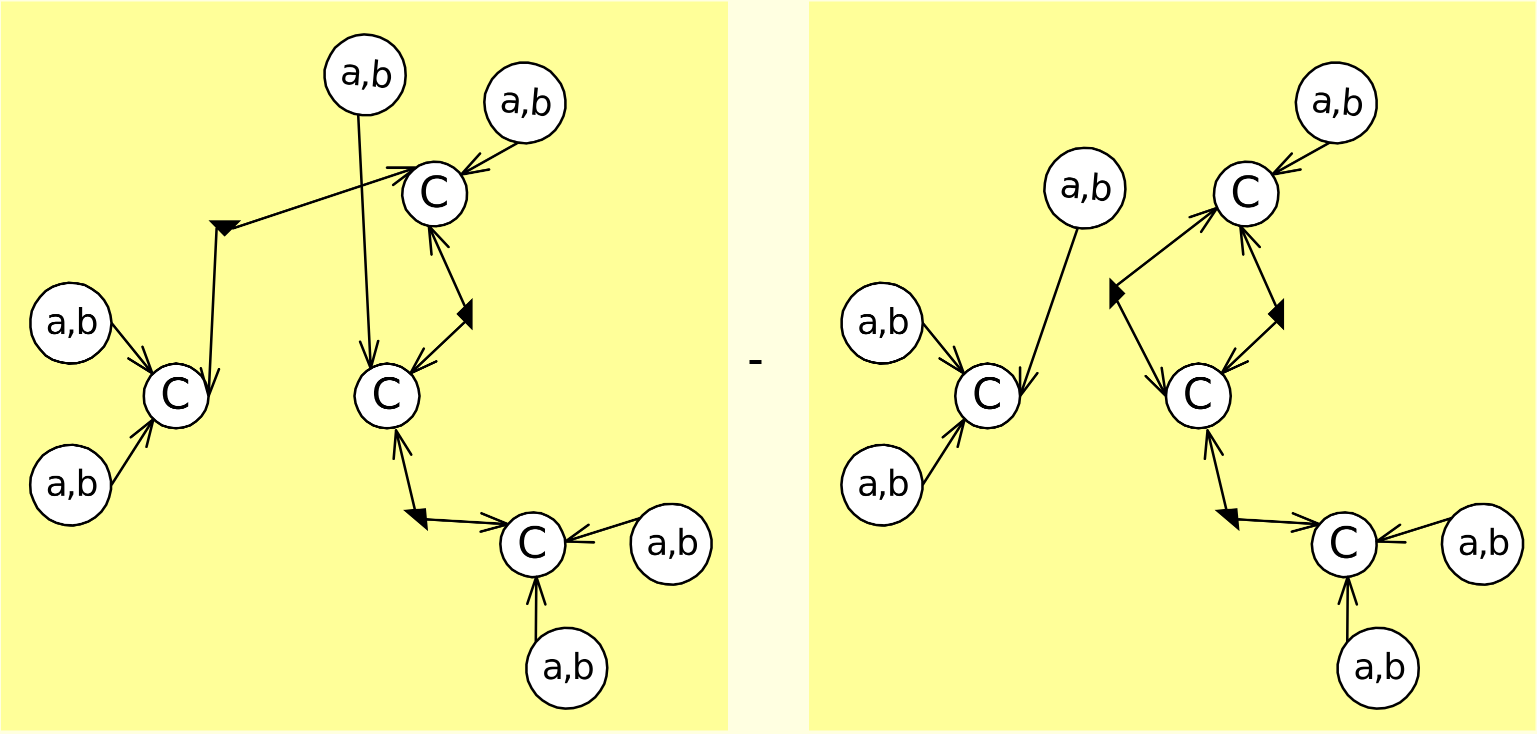
Evaluate D^

$\hat{D} =$



Evaluate D^

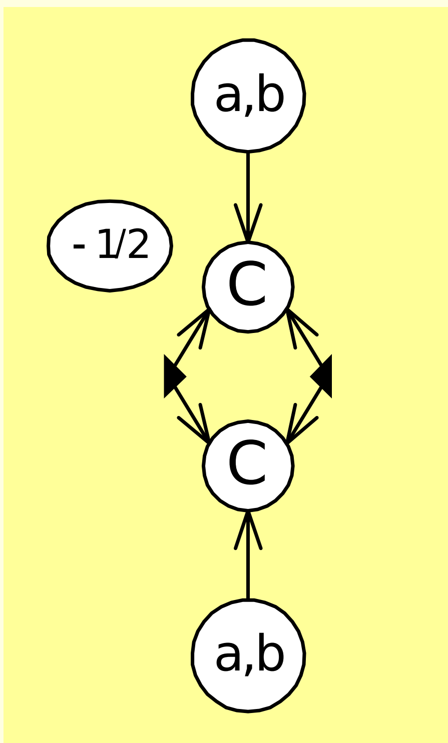
$\hat{D} =$



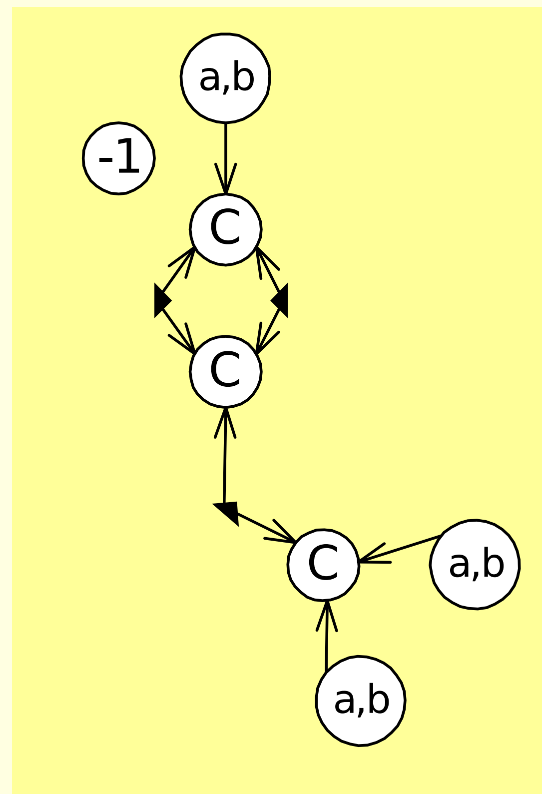
Transformed Cubic

$$\hat{A}\hat{x}^3 + 3\hat{C}\hat{x}\hat{w}^2 + \hat{D}\hat{w}^3 = \hat{A}\left(\hat{x}^3 + 3\tilde{C}\hat{x}\hat{w}^2 + \tilde{D}\hat{w}^3\right)$$

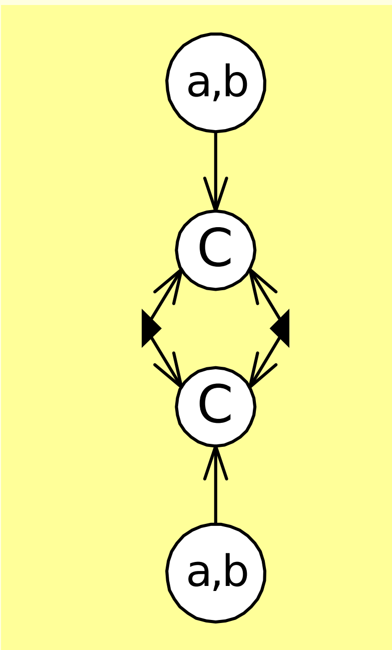
$\tilde{C} =$



$\tilde{D} =$



An Interesting Choice for a,b



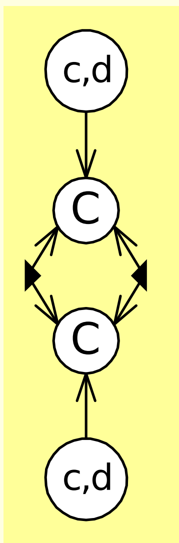
$=0$

$$\hat{A}\left(\hat{x}^3 + 3C\%\hat{x}\hat{w}^2 + D\%\hat{w}^3\right) = 0$$

ß

$$\hat{x}^3 + D\%\hat{w}^3 = 0$$

Implications for value of c,d



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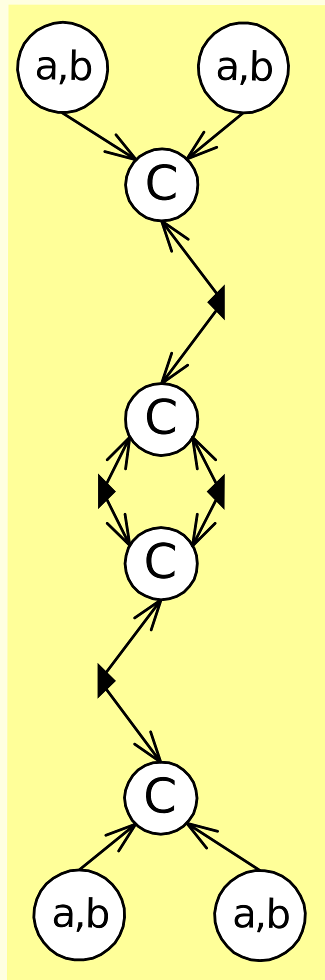

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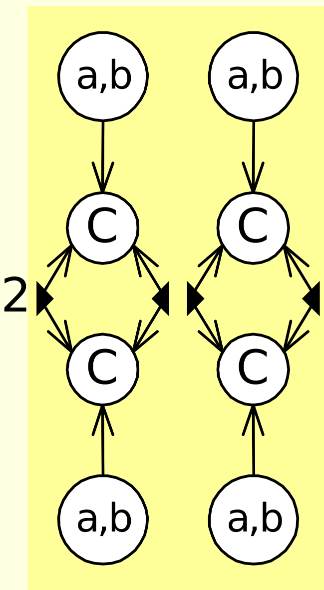
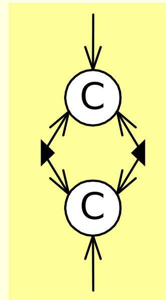


Solution

1) Find Roots of

$$=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

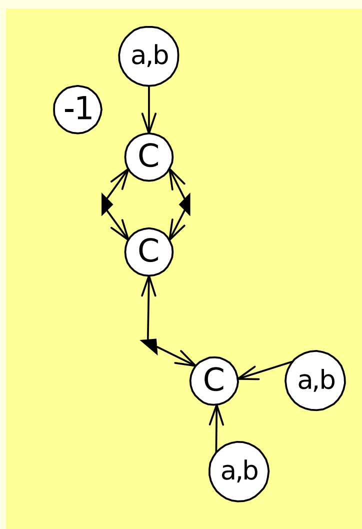


3) Solve for x^

$$\hat{x}^3 + \tilde{D}\hat{w}^3 = 0$$

2) Calculate

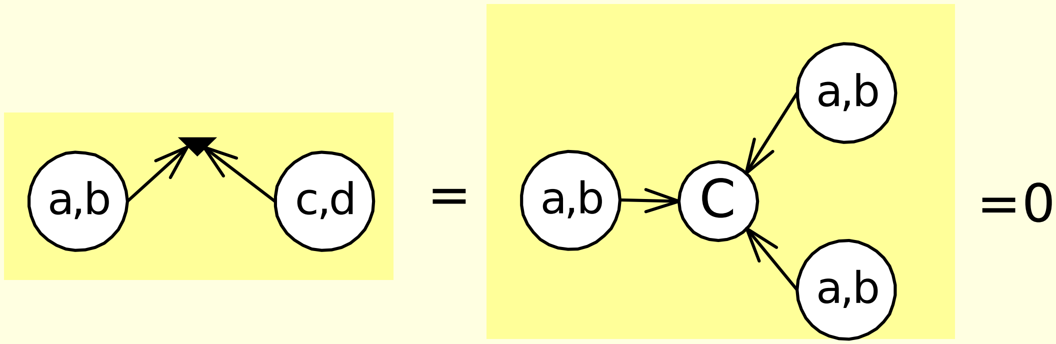
$\tilde{D} =$



4) Transform back via

$$[x \quad w] = [\hat{x} \quad \hat{w}]\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

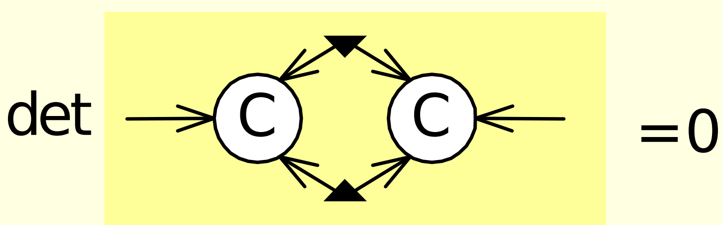
Only Time This Won’t Work

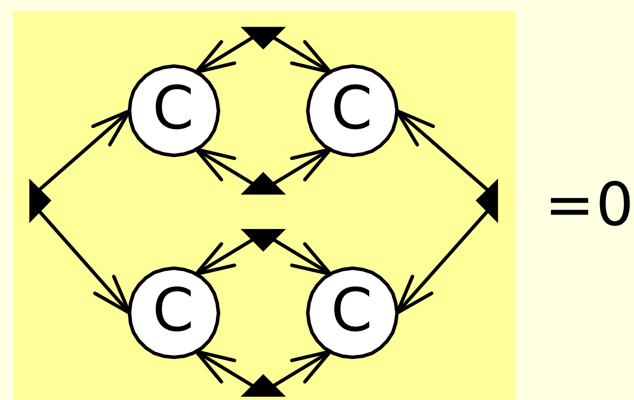


(a,b)=(c,d) is a double root of Quadratic

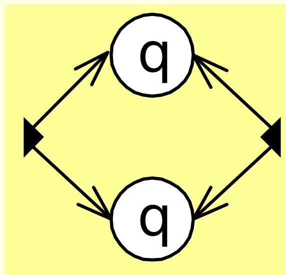
(a,b) is a root of C

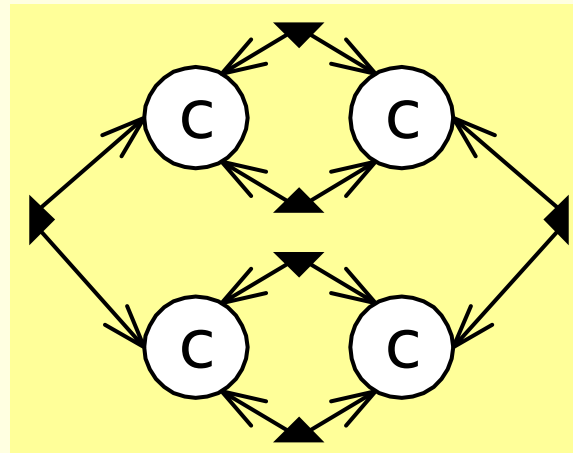
It is a double root of C

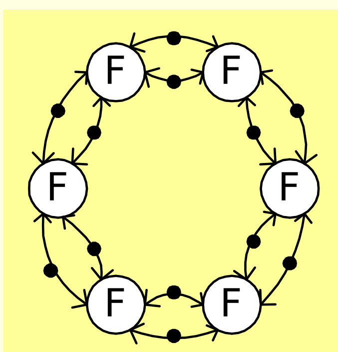




1DH Discriminants







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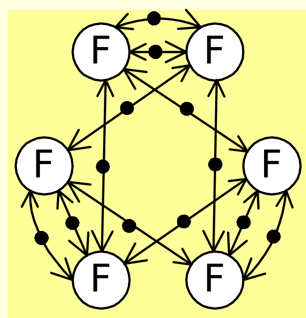

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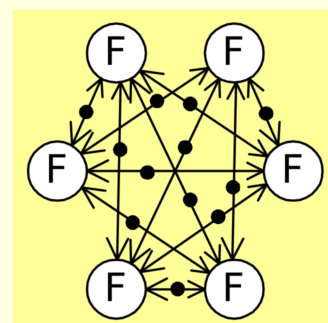
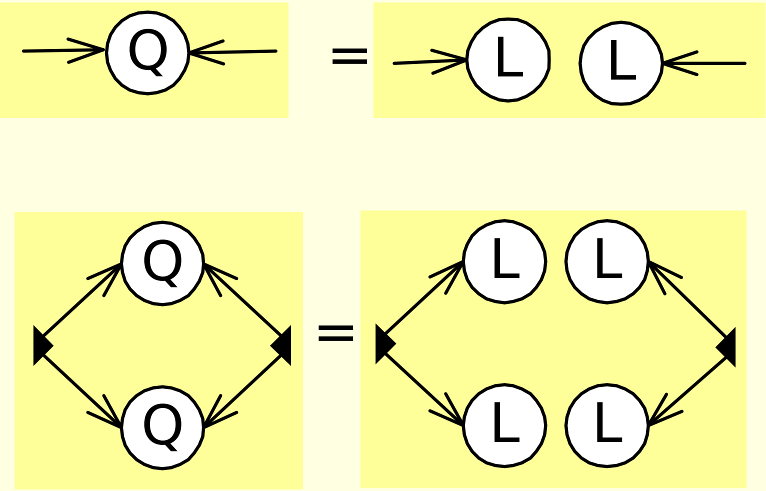
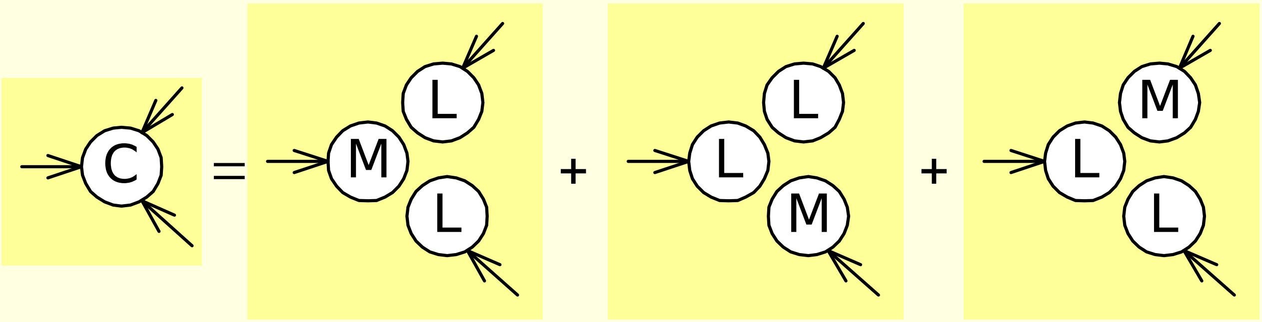
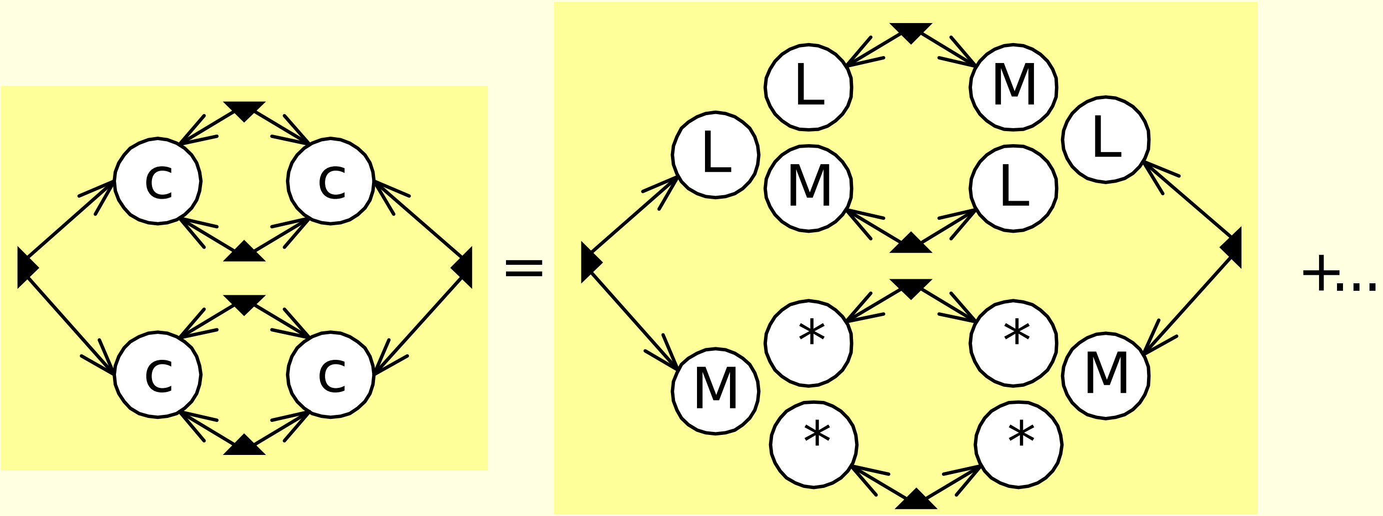


Why Discriminants Work

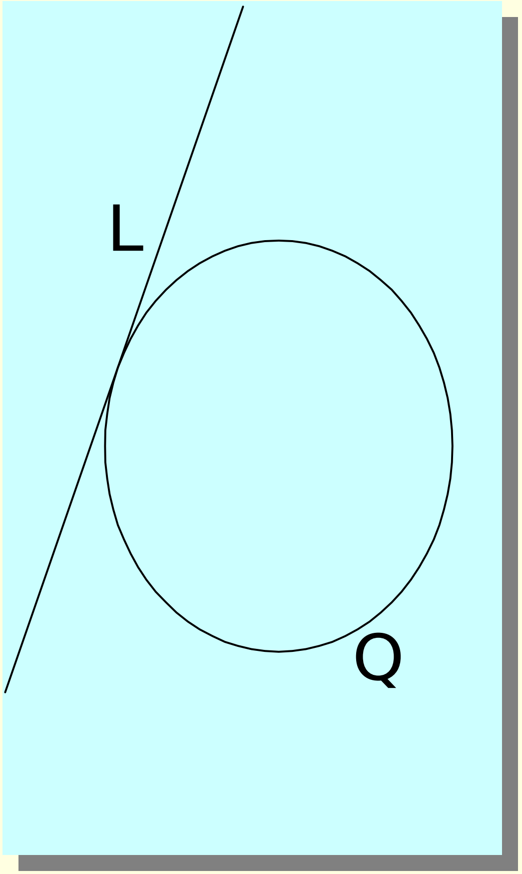


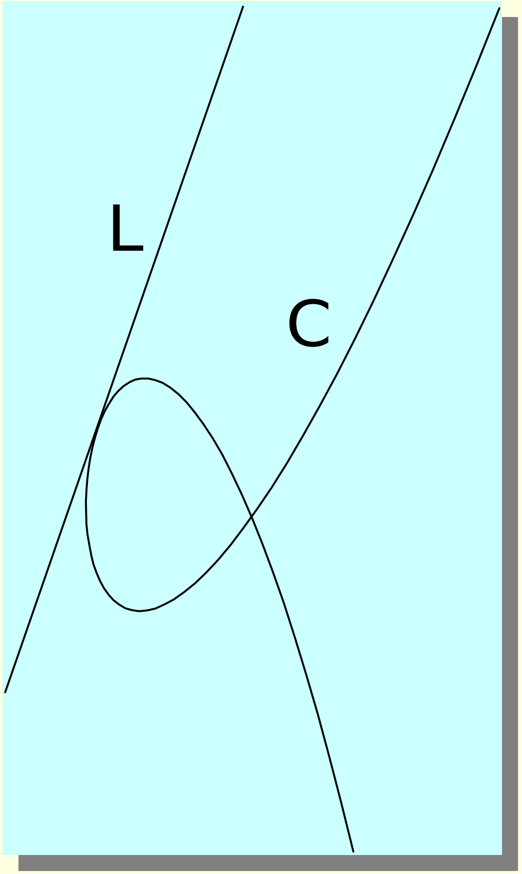
Why Discriminants Work





Tangency



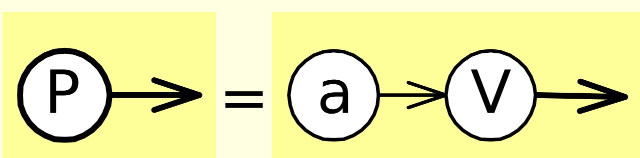


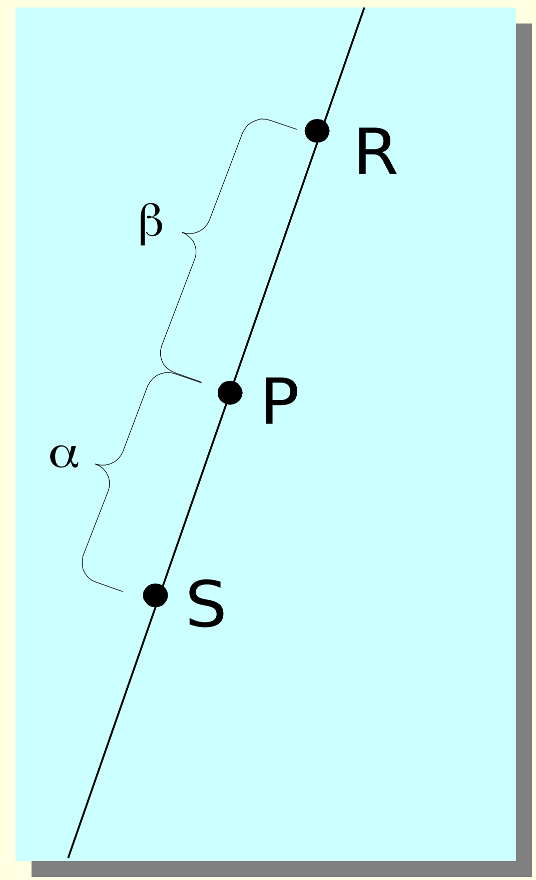
Parametrize Line

$$\mathbf{P}(a,b) = a\mathbf{R} + b\mathbf{S}$$

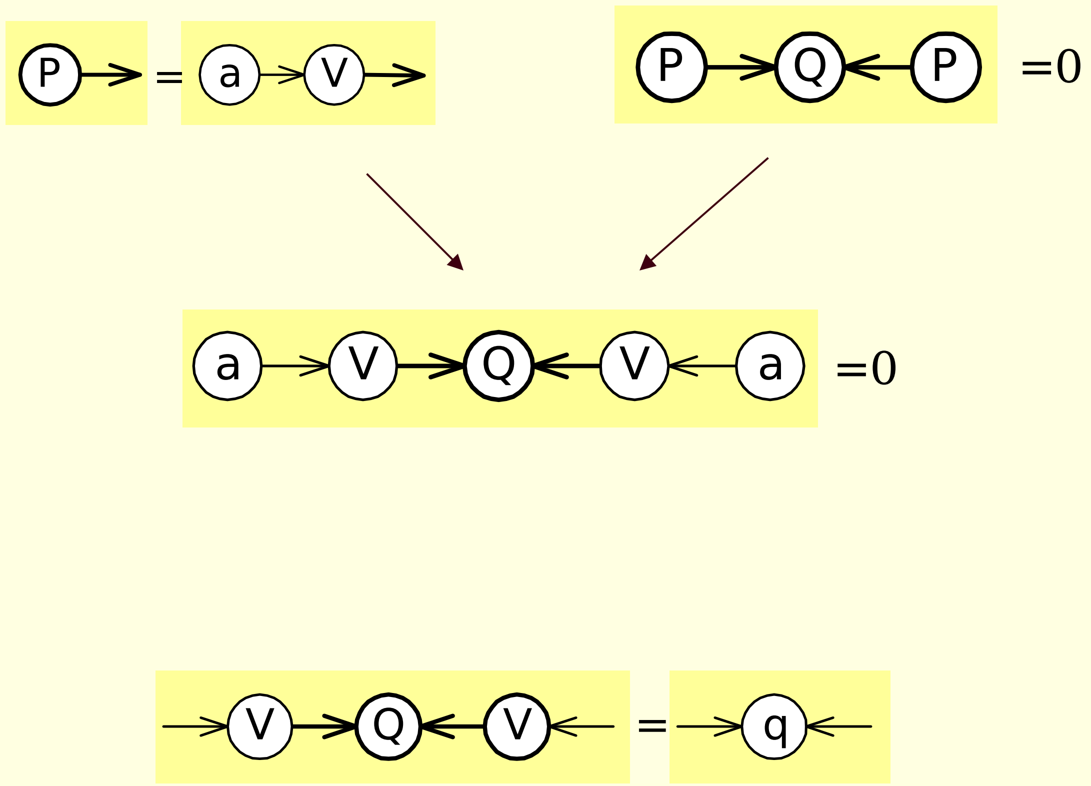
$$\mathbf{P} = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} R^1 & R^2 & R^3 \\ S^1 & S^2 & S^3 \end{bmatrix}$$

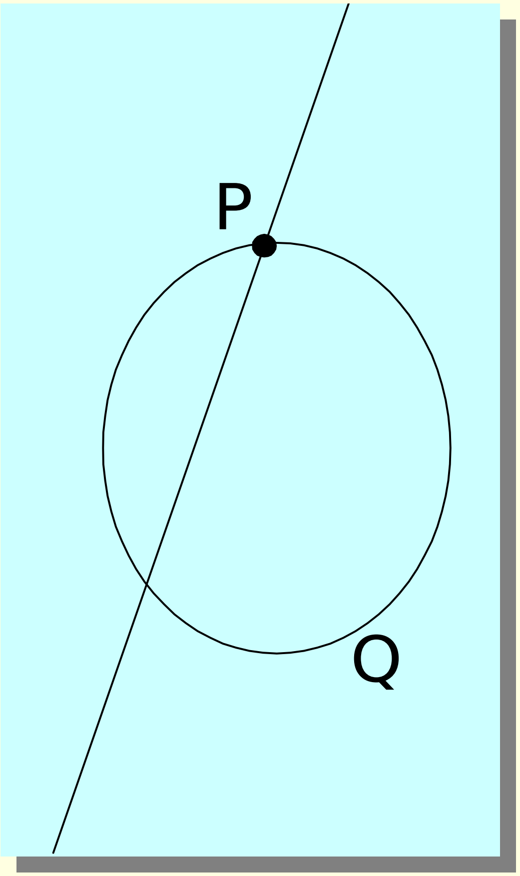
$$\mathbf{P} = \mathbf{aV}$$



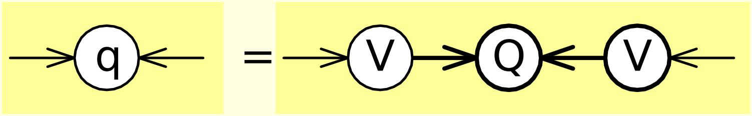


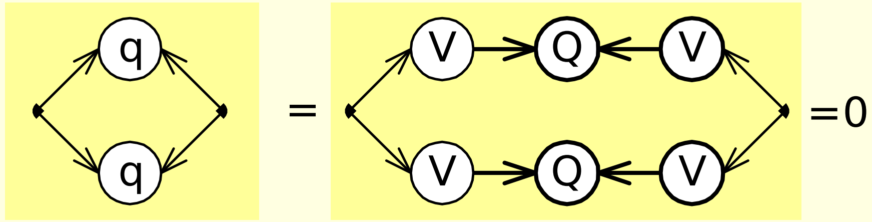
Points on Line And Quadric

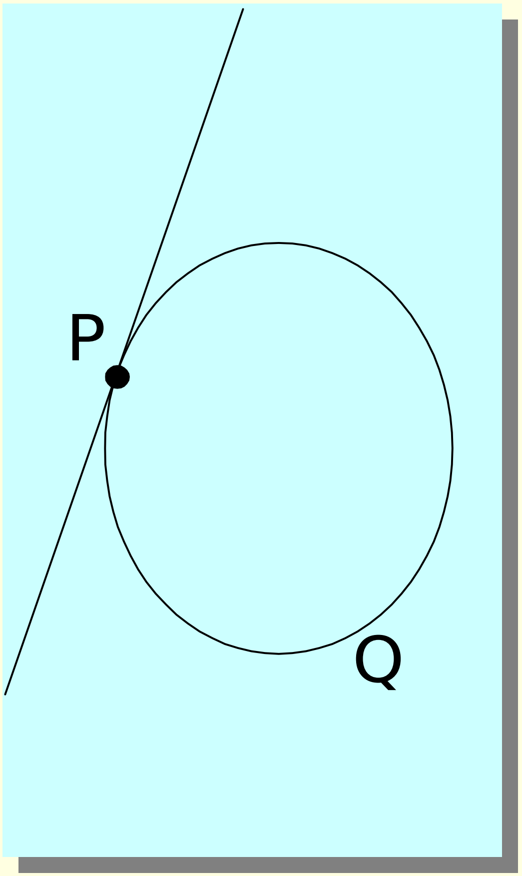




Double Roots Mean Tangent

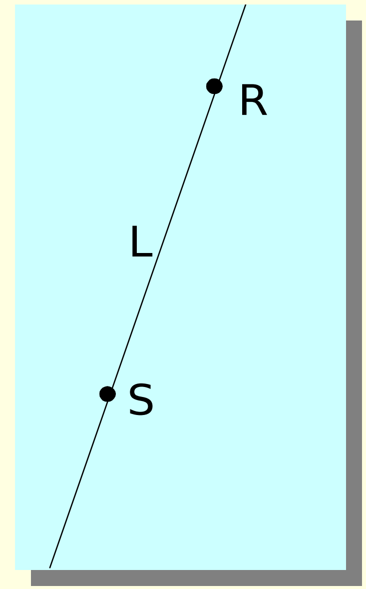


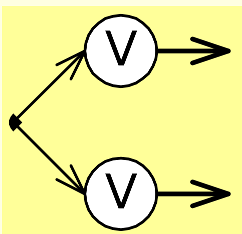




Reinterpret Diagram Fragment

$$\mathbf{V} = \begin{bmatrix} R^1 & R^2 & R^3 \\ S^1 & S^2 & S^3 \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$$



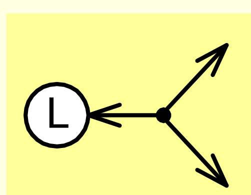


$$= \begin{bmatrix} R^1 & S^1 \\ R^2 & S^2 \\ R^3 & S^3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} R^1 & R^2 & R^3 \\ S^1 & S^2 & S^3 \end{bmatrix}$$

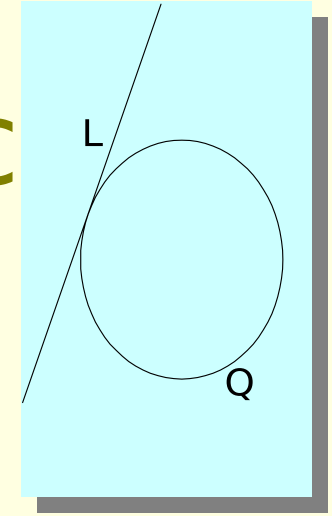
$$= \begin{bmatrix} 0 & R^1S^2 - R^2S^1 & R^1S^3 - R^3S^1 \\ R^2S^1 - R^1S^2 & 0 & R^2S^3 - R^3S^2 \\ R^3S^1 - R^1S^3 & R^3S^2 - R^2S^3 & 0 \end{bmatrix}$$

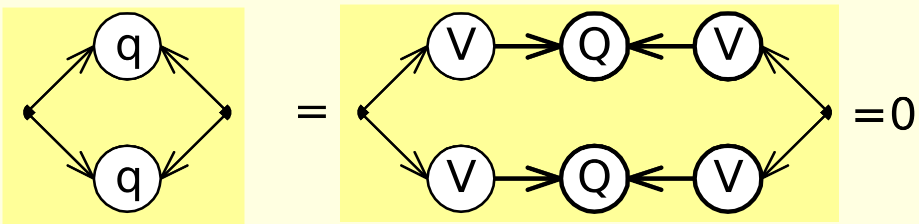
$$= \begin{bmatrix} 0 & L_3 & -L_2 \\ -L_3 & 0 & L_1 \\ L_2 & -L_1 & 0 \end{bmatrix}$$

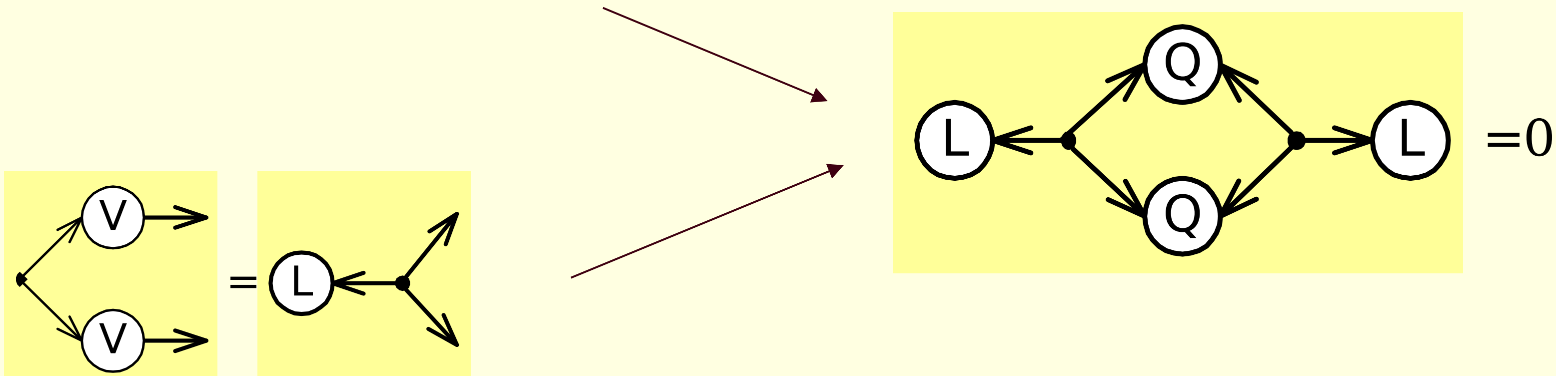
=

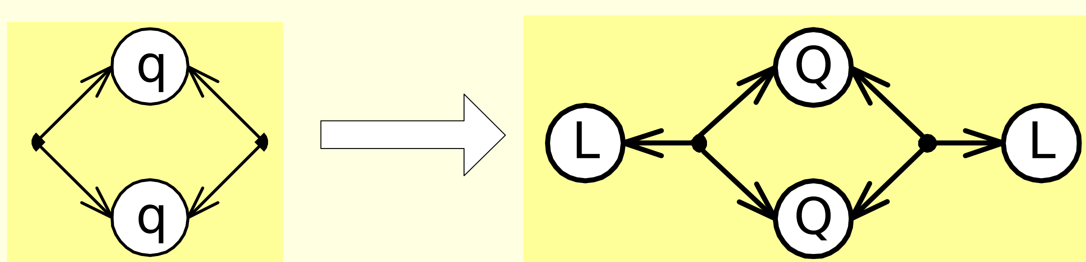


Line Tangent To Quadric

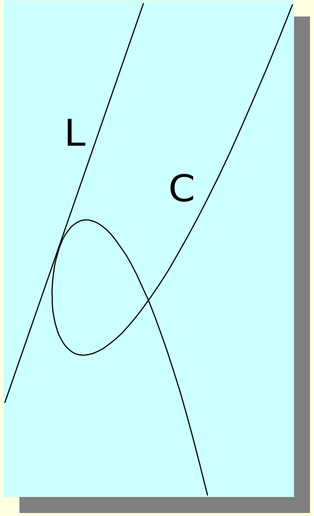


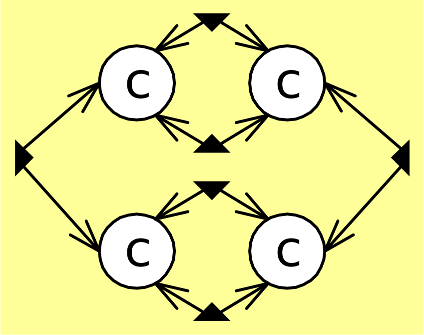






Line Tangent to Cubic





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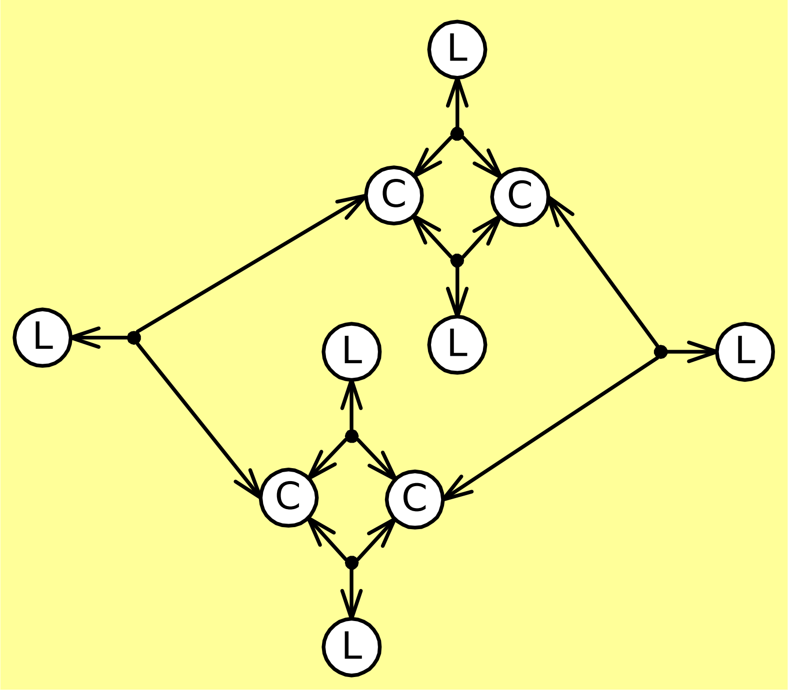


=0

Polynomials

Resultants

Resultant of Two Polynomials

$$Q(X) = AX^2 + 2BX + C$$

$$R(X) = DX^2 + 2EX + F$$

$$\mathbf{R}(Q,R) = f(A,B,C,D,E,F)$$

$$\mathbf{R} = 0 \quad \text{Û} \quad Q \text{ and } R \text{ have a common root}$$

Calculating the Resultant

$$Q(X) = AX^2 + 2BX + C = 0$$

$$R(X) = DX^2 + 2EX + F = 0$$

$$aQ + bR = 0$$

$$DQ - AR = 0$$

$$D\left(AX^2 + 2BX + C\right)$$

$$-A\left(DX^2 + 2EX + F\right) = 0$$

$$2(BD - AE)X + (CD - AF) = 0$$

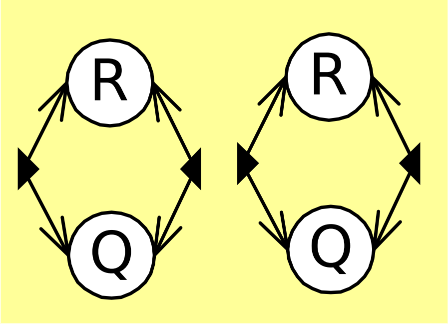
Resultant of Q and R

$$Q(x,w) = Ax^2 + 2Bxw + Cw^2$$

$$R(x,w) = Dx^2 + 2Exw + Fw^2$$

$$\mathbf{R}(Q,R) = +A^2F^2 - 4ABEF + 4ACE^2 - 2ACDF$$
$$+4B^2DF - 4BCED + C^2D^2$$

$\mathbf{R}(Q,R) =$



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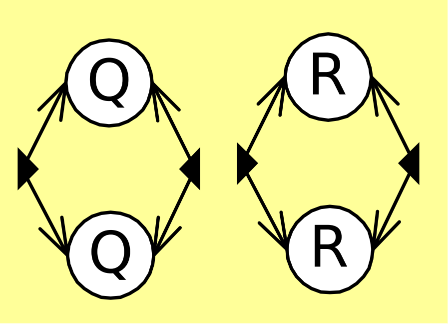
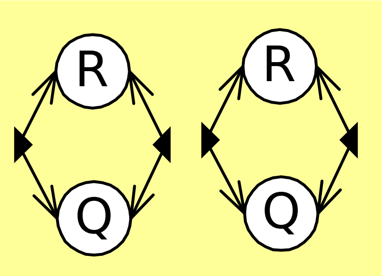
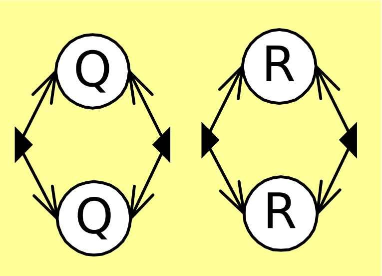
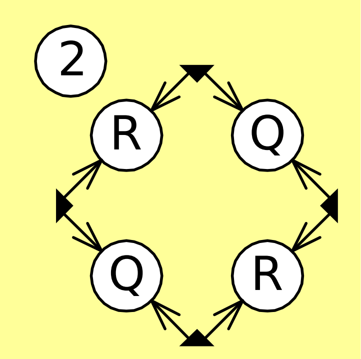
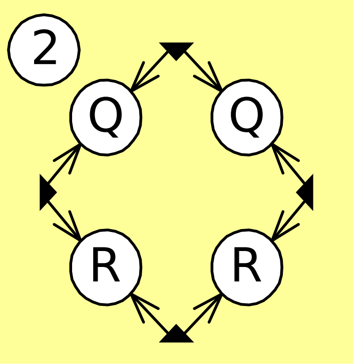


Two Forms of Resultant

$\mathbf{R}(Q,R) =$ 

- 

$\mathbf{R}(Q,R) =$ 

- 

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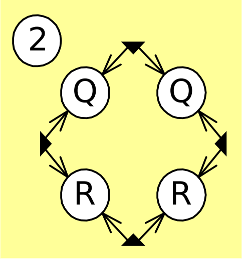
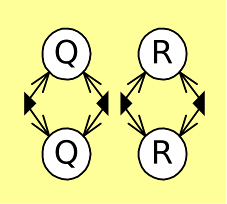
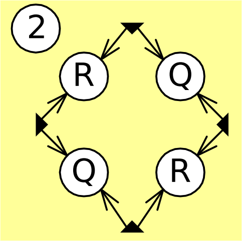
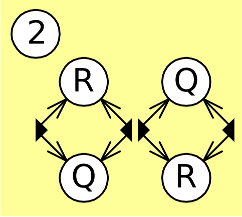
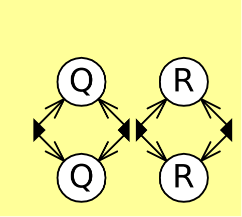


= 



= 

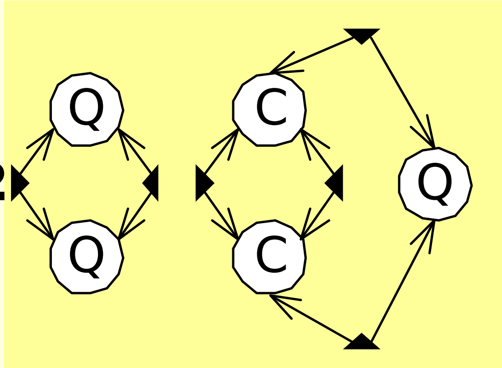
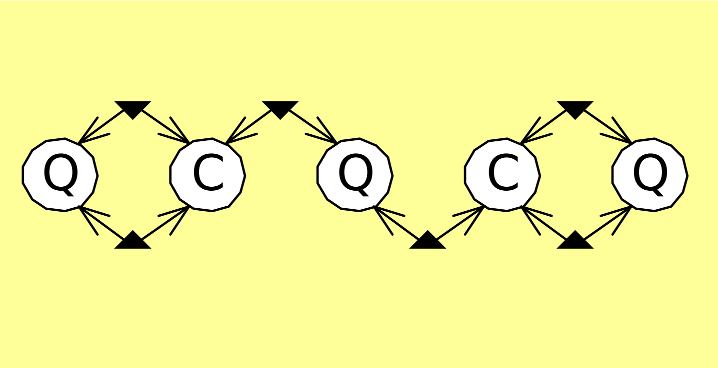
- 

Resultant of Q and C

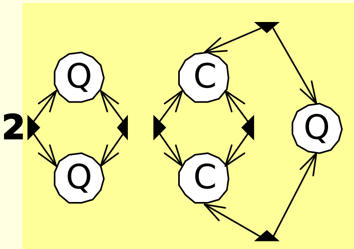
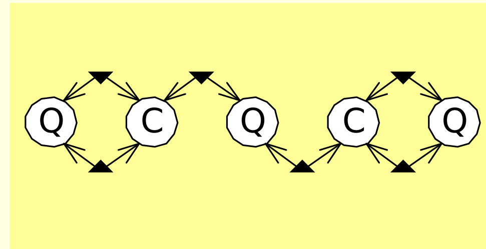
$$\mathbf{C}(x,w) = Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3$$

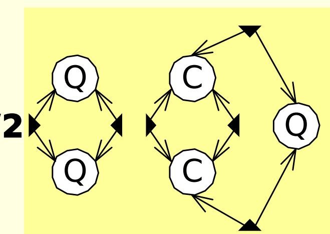
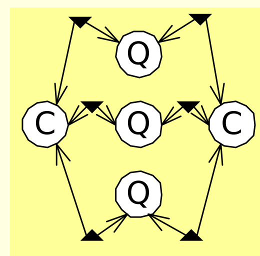
$$\mathbf{Q}(x,w) = Ex^2 + 2Fxw + Gw^2$$

$$\begin{aligned}\mathbf{R}(\mathbf{Q},\mathbf{C}) = &- A^2(G^3) + 6AB(FG^2) - 6AC(2F^2G - EG^2) - 2AD(-4F^3 + 3EFG)\\ &- 9B^2(EG^2) + 18BC(EFG) - 6BD(2EF^2 - E^2G)\\ &- 9C^2(E^2G) + 6CD(E^2F)\\ &- D^2(E^3)\end{aligned}$$

R(Q,C) = - 2  - 

Forms of Q,C Resultant

R(Q,C) = - 2  - 

R(Q,C) = - 3/2  + 

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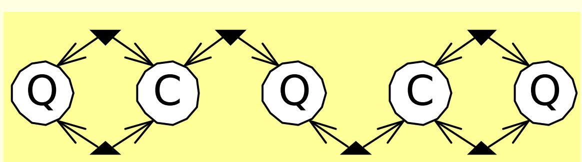
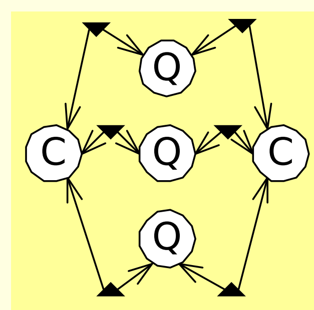
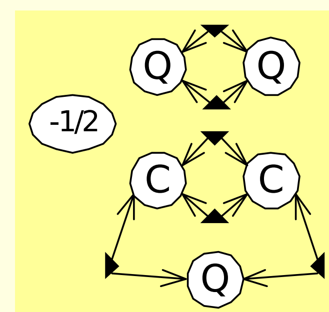
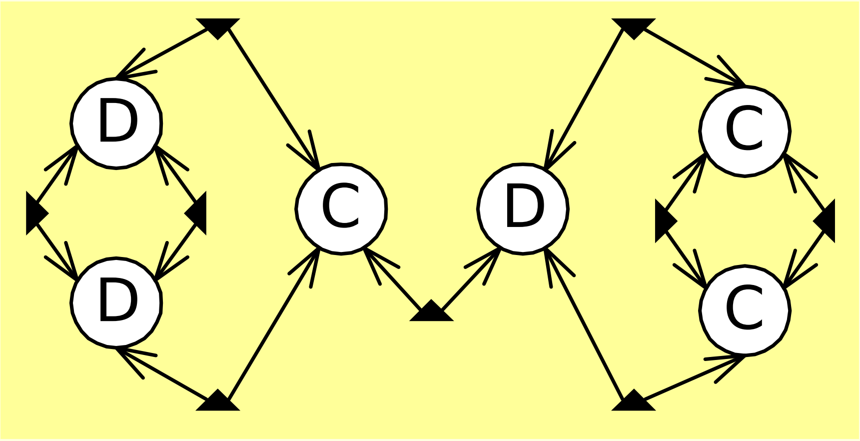
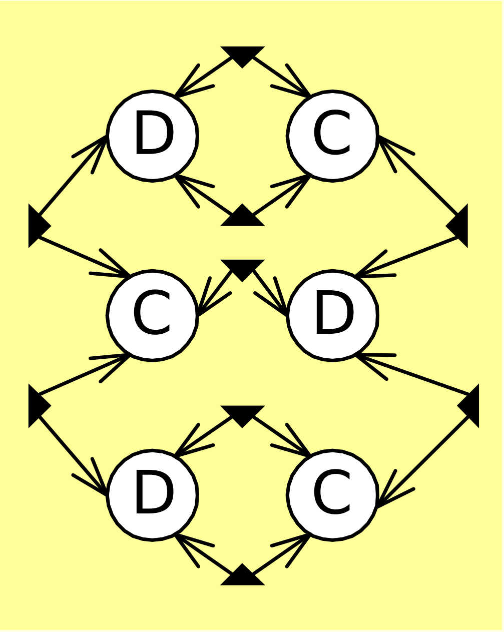
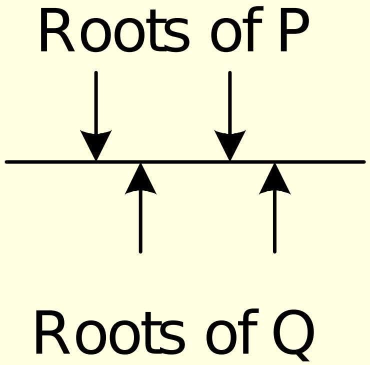
 = -  + 

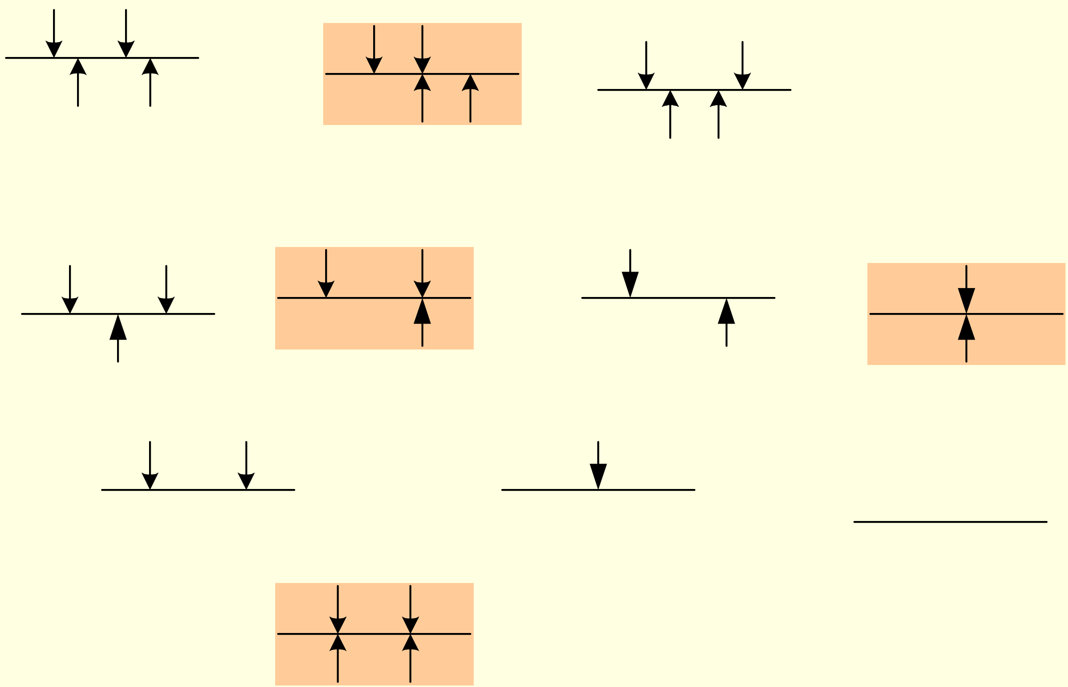
Resultant of two Cubics ?

2  - 

What’s Really Going on With Resultants?

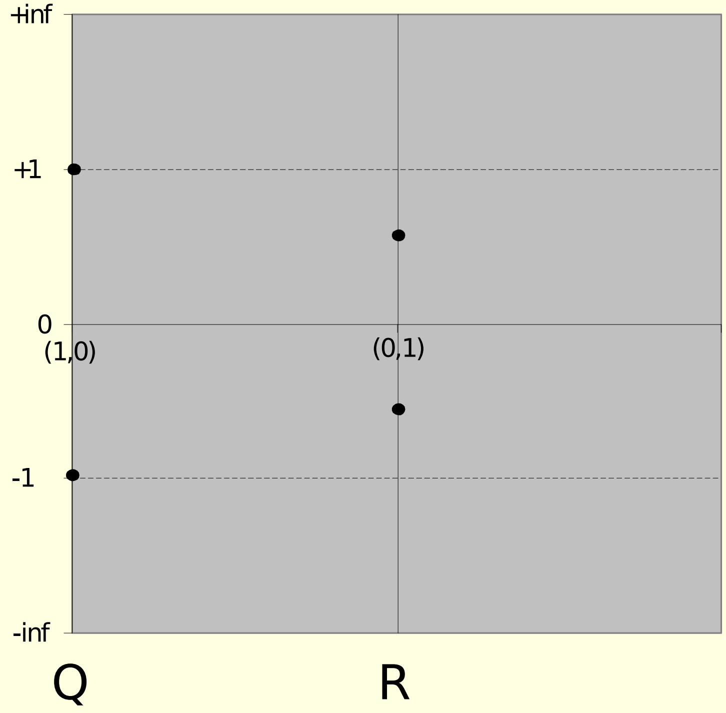
Possible Relationships between P and Q





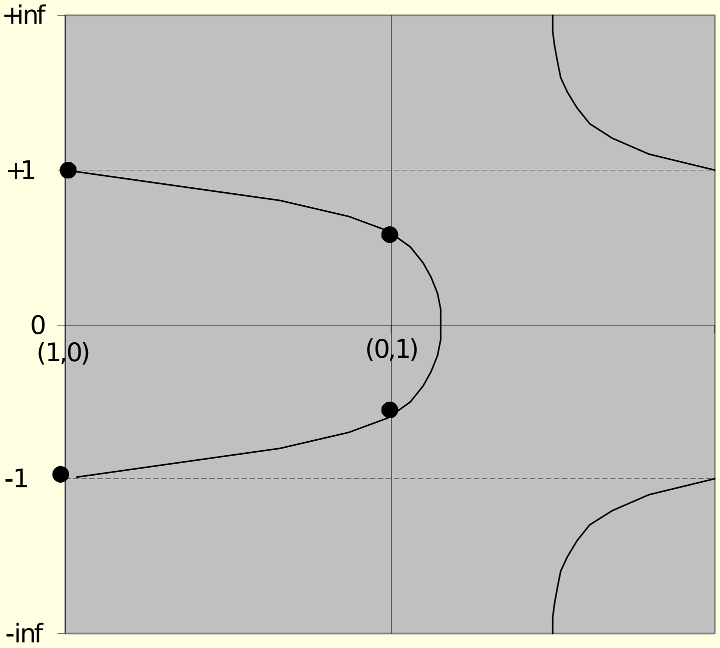
Linear Combos of Q and R

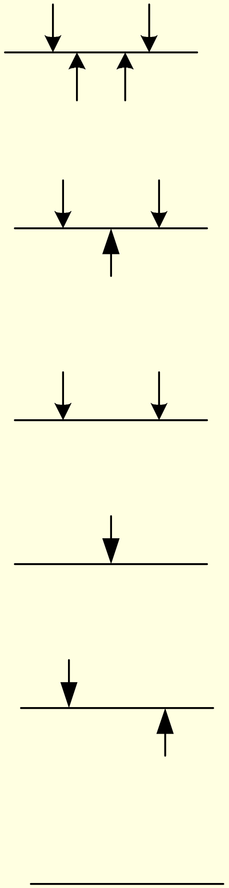
$$aQ + bR = 0$$



Q,R Have Enclosed Roots

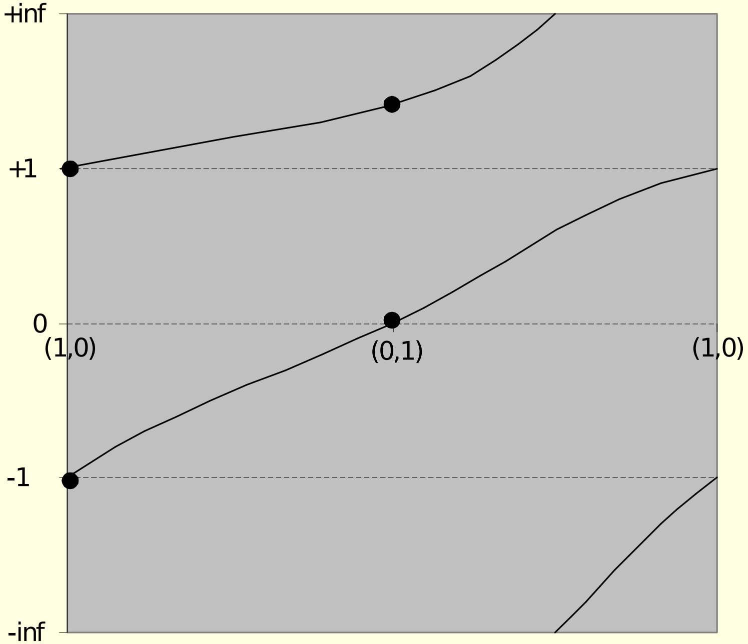
$$aQ + bR = 0$$

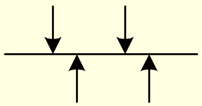




Q,R Have Interleaved Roots

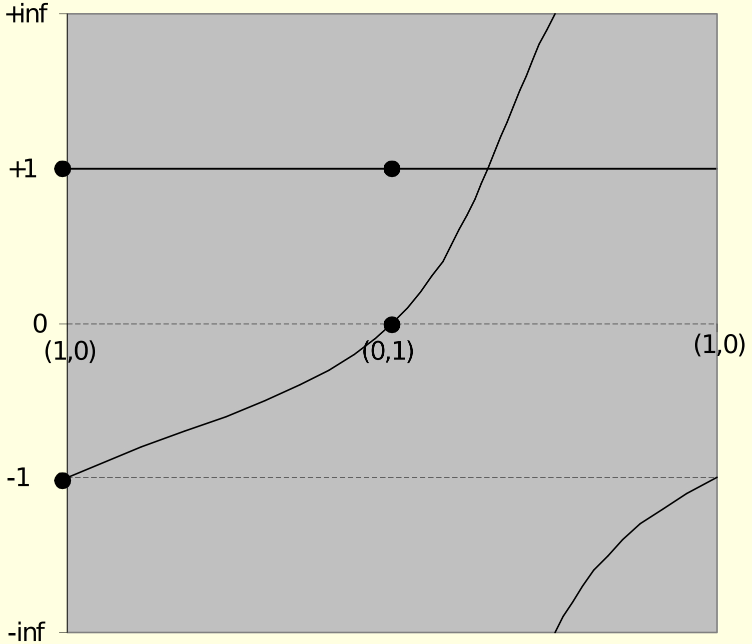
$$aQ + bR = 0$$

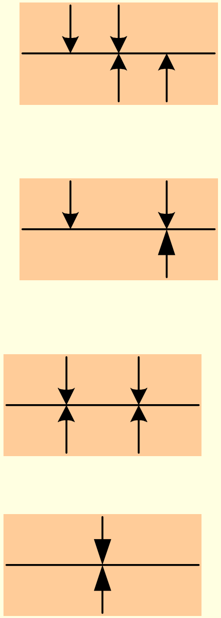




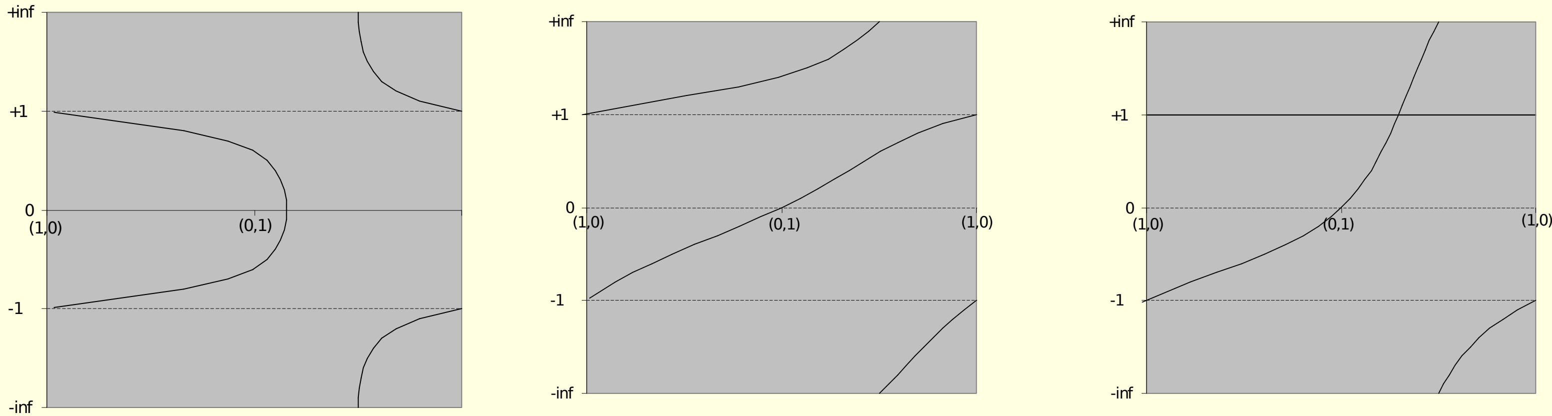
Q,R Have Common Root

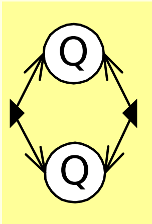
$$aQ + bR = 0$$

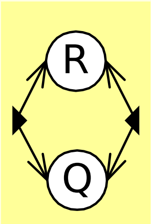


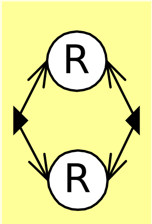


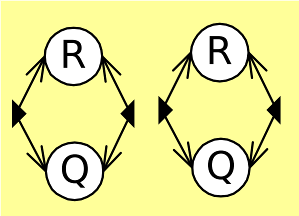
Three Possible Evolutions of Roots of $aQ+bR$

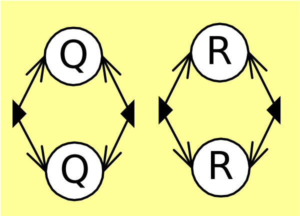


$\det(aQ+bR) =$ α^2 

$+2\alpha\beta$ 

$+\beta^{2}$ 

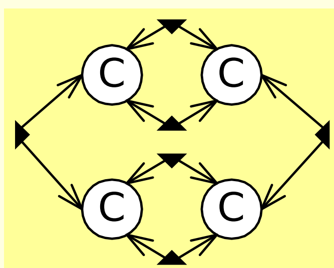
$\mathbf{R}(Q,R) =$ 

\- 

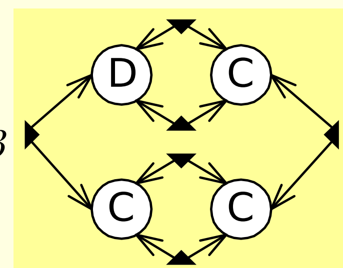
Possible Evolutions of Roots of Two Cubics

$$\det\left(a\mathbf{C}+b\mathbf{D}\right) =$$

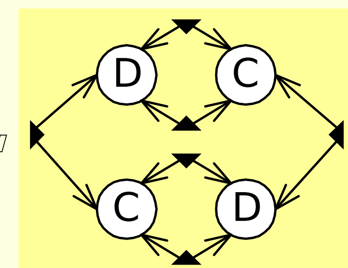
α^{4}



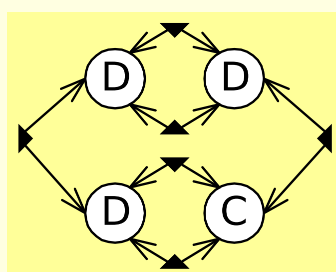
$+4\alpha^{3}\beta$



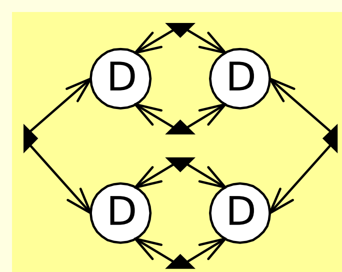
$+\square\,\alpha^{\square}\beta^{\square}$



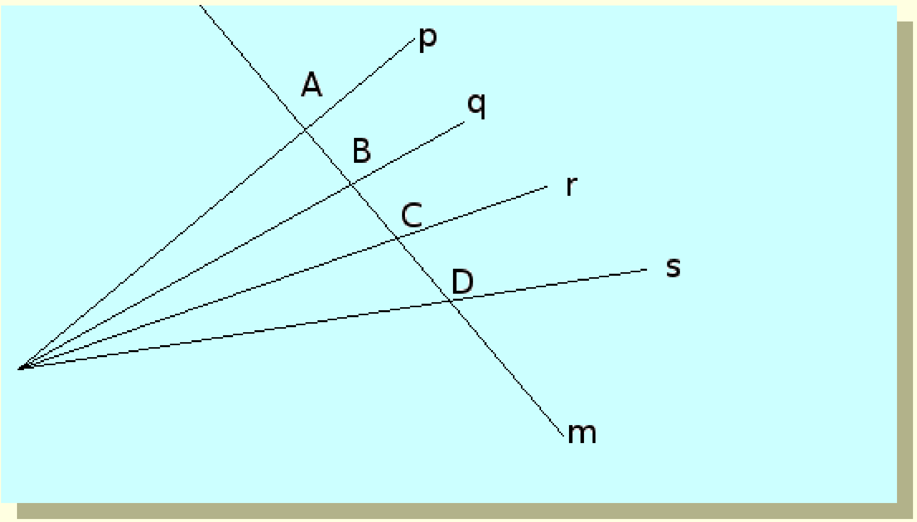
$+\square\,\alpha\beta^{\square}$



$+\beta^{\square}$



The Cross Ratio

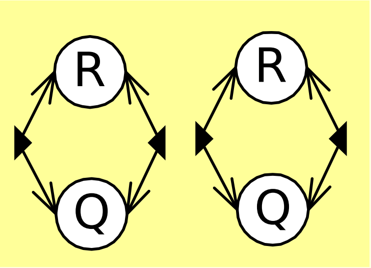


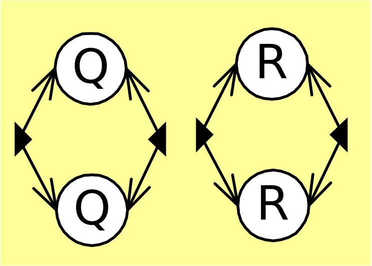
$$\chi = \frac{|\mathbf{AB}|/|\mathbf{BD}|}{|\mathbf{AC}|/|\mathbf{CD}|}$$

$$c = \frac{(\mathbf{p} \times \mathbf{q} \cdot \mathbf{m})(\mathbf{r} \times \mathbf{s} \cdot \mathbf{m})}{(\mathbf{q} \times \mathbf{s} \cdot \mathbf{m})(\mathbf{p} \times \mathbf{r} \cdot \mathbf{m})}$$

Generalized Cross Ratio of Two Quadratic Polynomials

$\chi =$





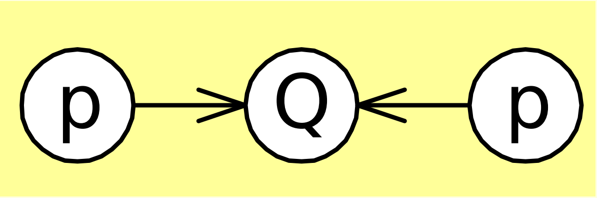
2DH Curves

Quadratic

The Quadratic Curve Equation

$$Ax^2 + 2Bxy + Cy^2$$
$$+ 2Dxw + 2Eyw$$
$$+ Fw^2 = 0$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} A & B & D \\ B & C & E \\ D & E & F \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \mathbf{pQp}^T = 0$$



$= 0$

Transform to

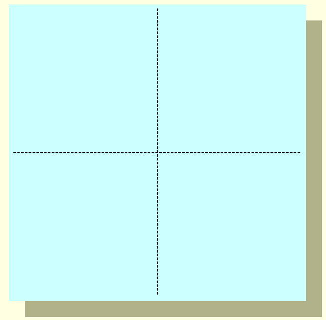
$$\mathbf{T}^{*}\begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix}\mathbf{T}^{*T} = \begin{bmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & \lambda_w \end{bmatrix}$$

$$\begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & S_w \end{bmatrix}\begin{bmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & \lambda_w \end{bmatrix}\begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & S_w \end{bmatrix} = \begin{bmatrix} U_x & 0 & 0 \\ 0 & U_y & 0 \\ 0 & 0 & U_w \end{bmatrix}$$

$$U_i = -1, 0, +1$$

The Catalog

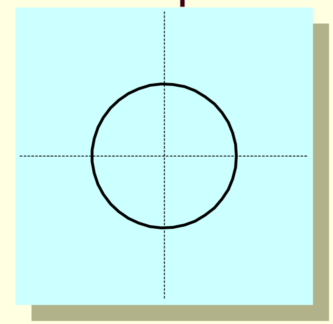
+++



$x^2 + y^2 + w^2 = 0$

++-

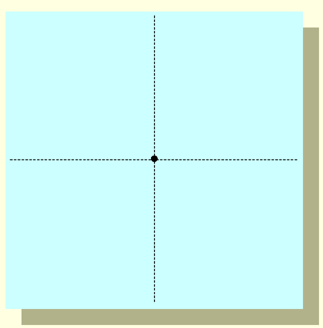
--+



$x^2 + y^2 - w^2 = 0$

++0

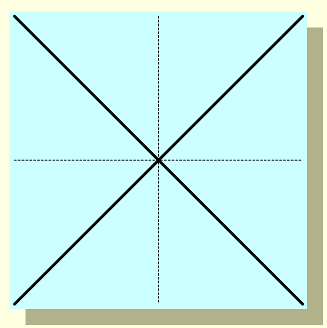
--0



$x^2 + y^2 = 0$

+-0

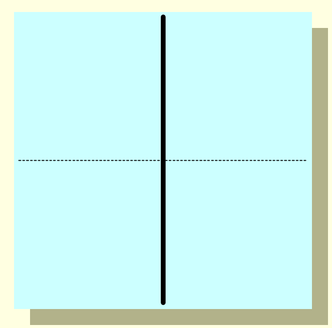
-+0



$x^2 - y^2 = 0$

+00

-00

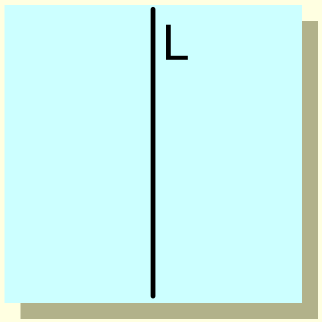


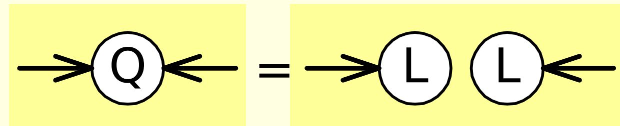
$x^2 = 0$

Analysis/Synthesis of Forms

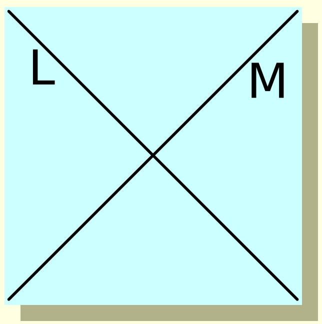
- Detect Which Type
- Construct Desired Type from Geometric Info
- Deconstruct Known Type into Geometric Info
- Stationary Transforms

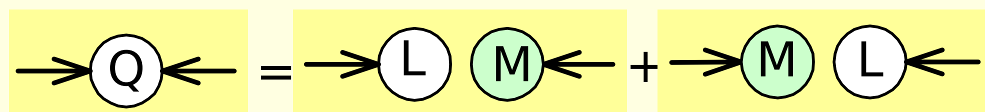
Reducible Quadratics



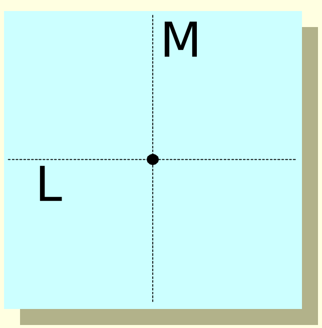


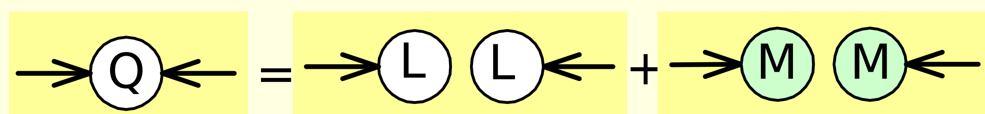
$$\mathbf{PQP}^T = \left(\mathbf{P}\times\mathbf{L}\right)^2$$





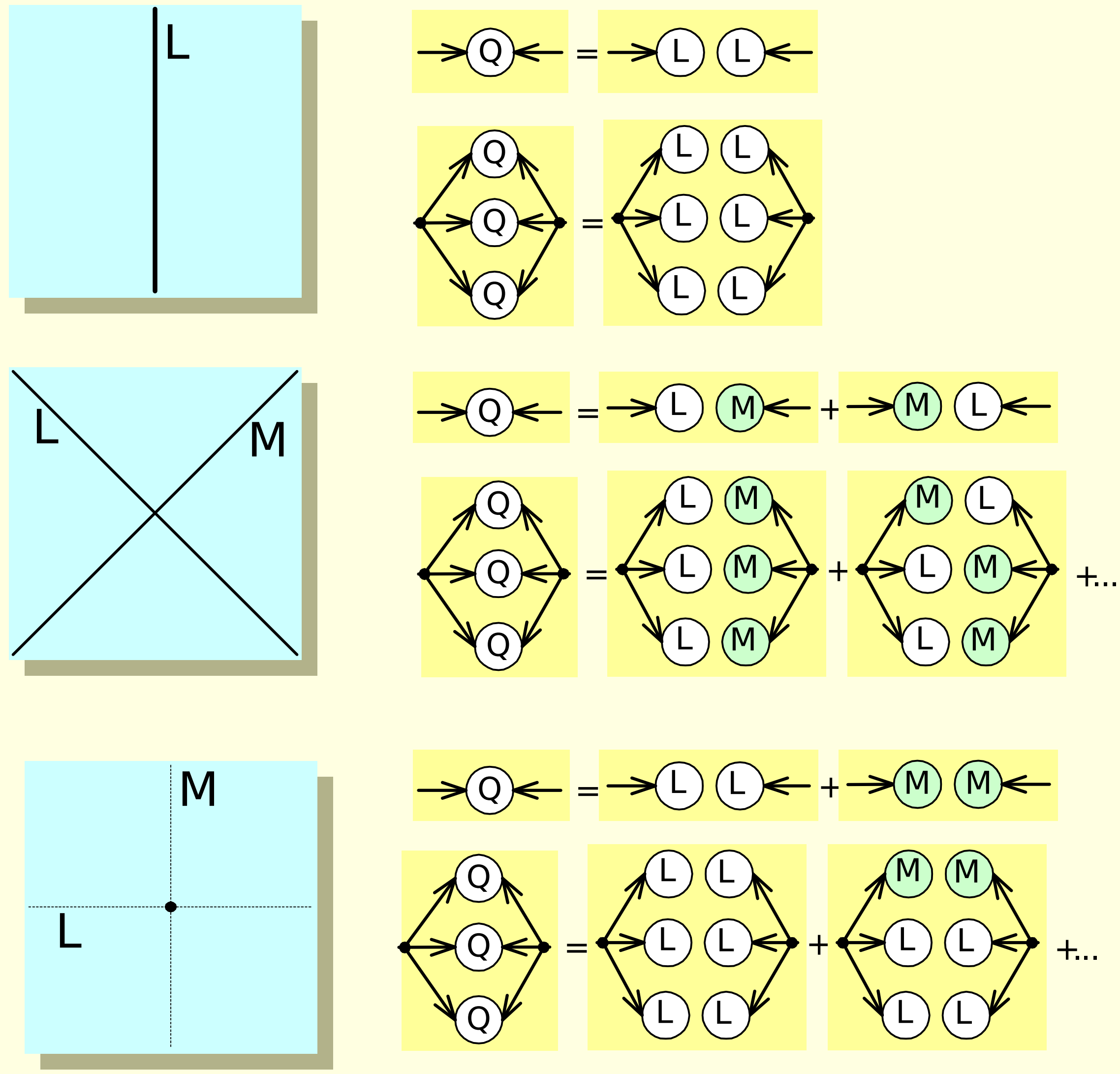
$$\mathbf{PQP}^T = 2\left(\mathbf{P}\times\mathbf{L}\right)\left(\mathbf{P}\times\mathbf{M}\right)$$



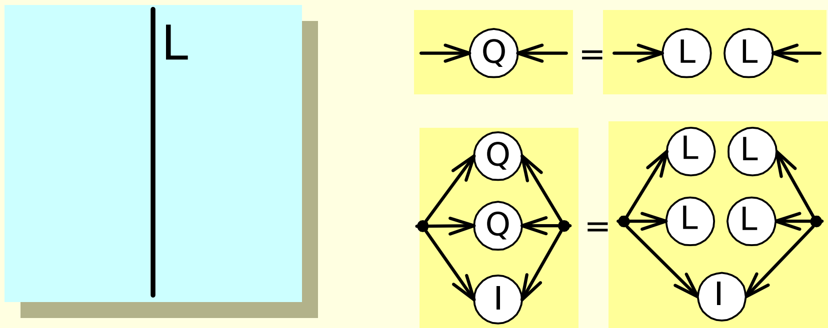


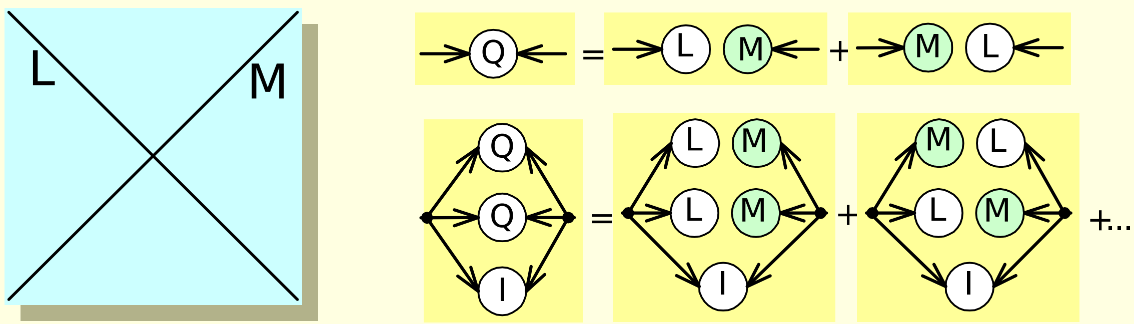
$$\mathbf{PQP}^T = \left(\mathbf{P}\times\mathbf{L}\right)^2 + \left(\mathbf{P}\times\mathbf{M}\right)^2$$

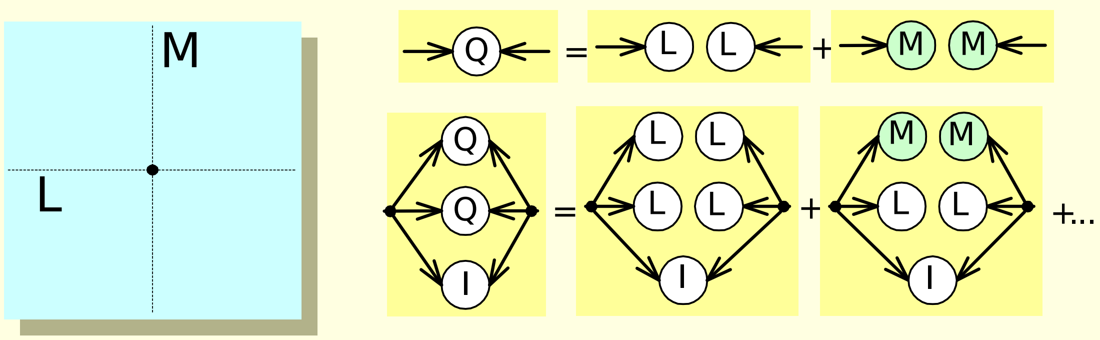
Determinant of Reducible Q



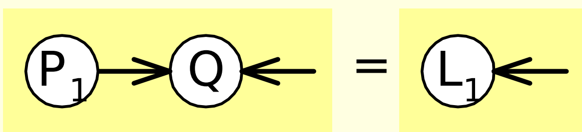
TraceAdjoint of Reducible Q

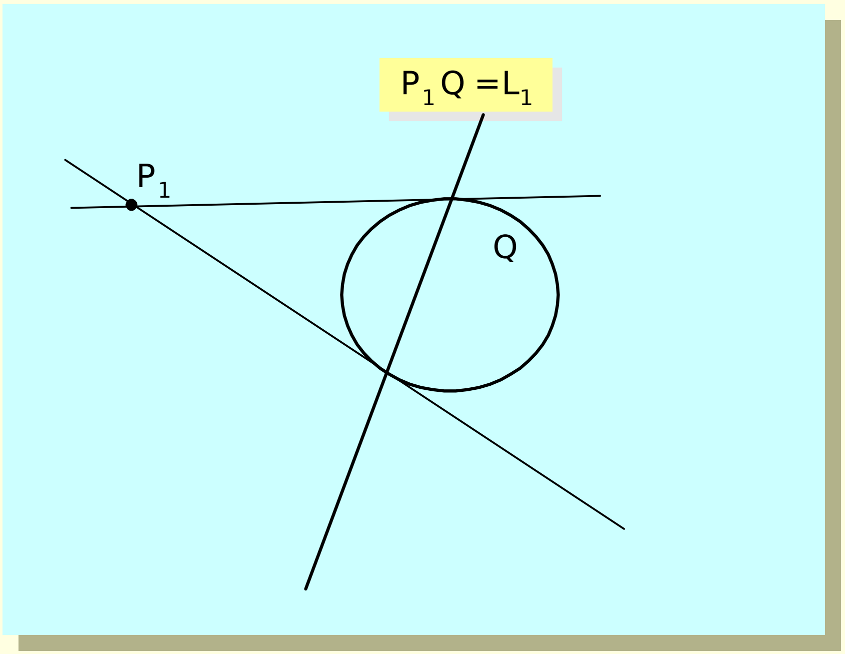




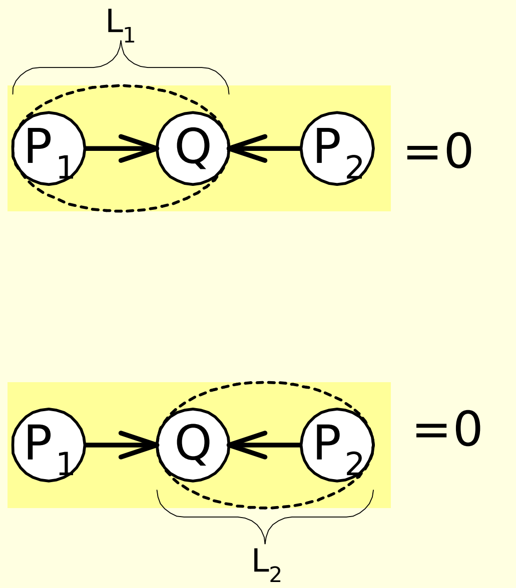


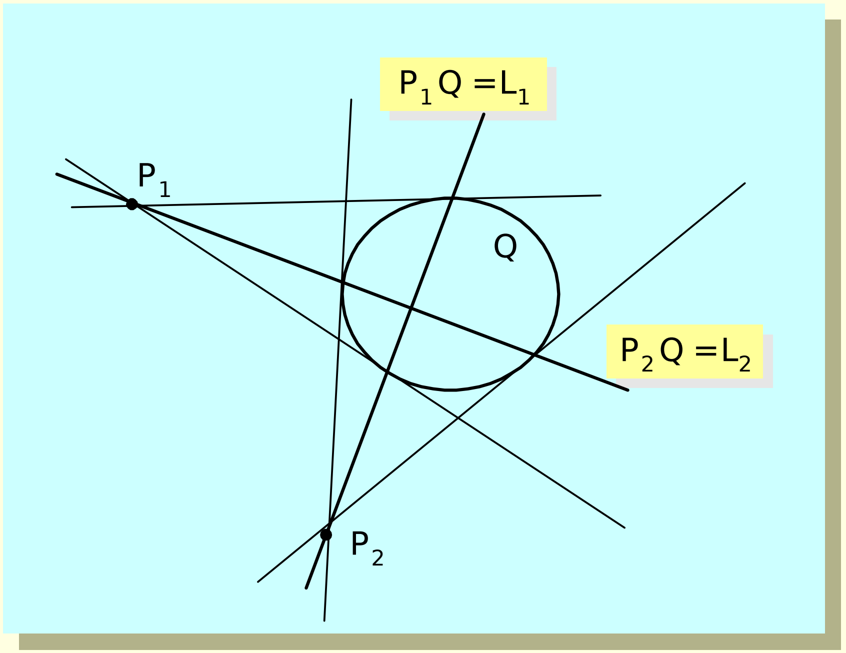
Conic Sections and Polars



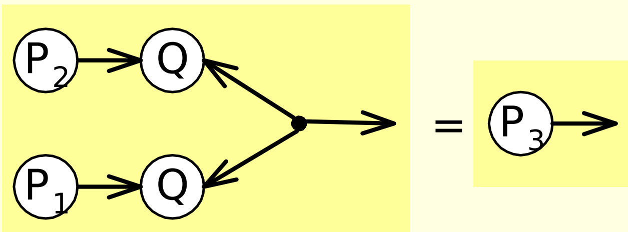


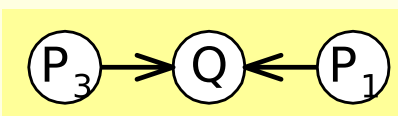
Second Polar





Third Polar





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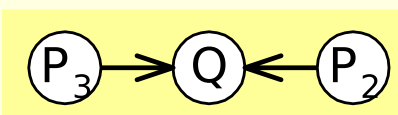

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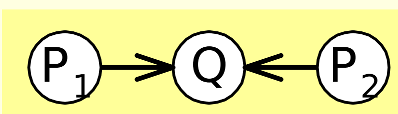

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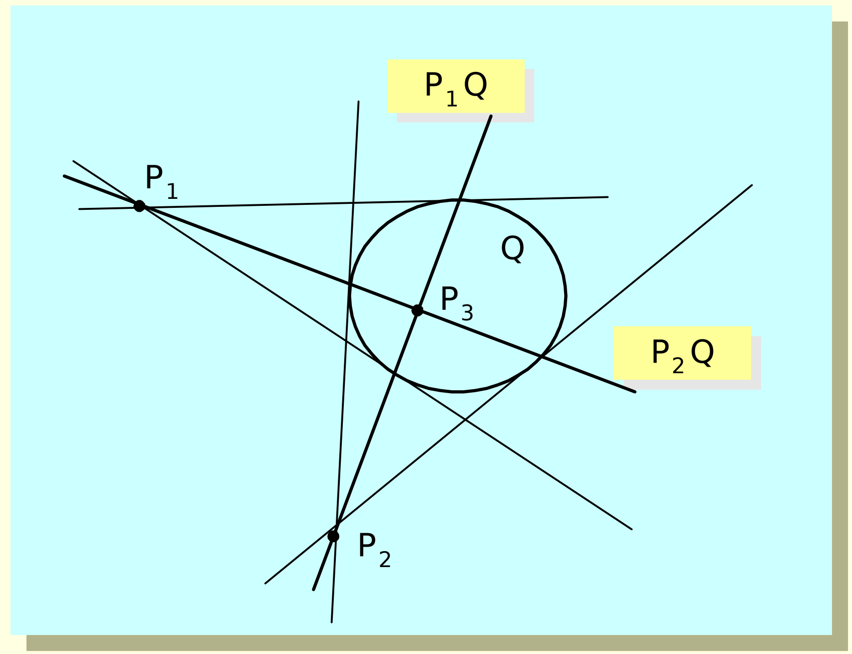
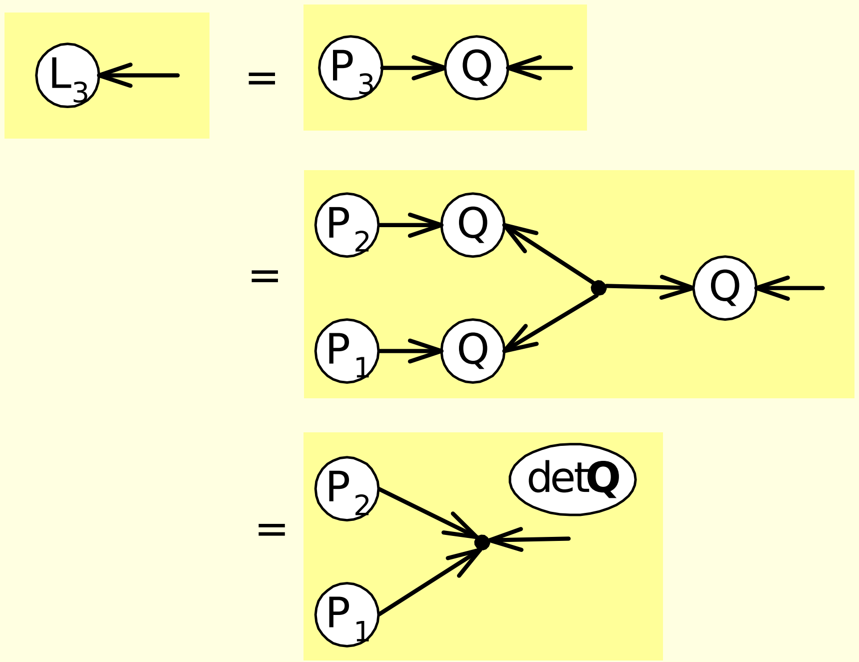
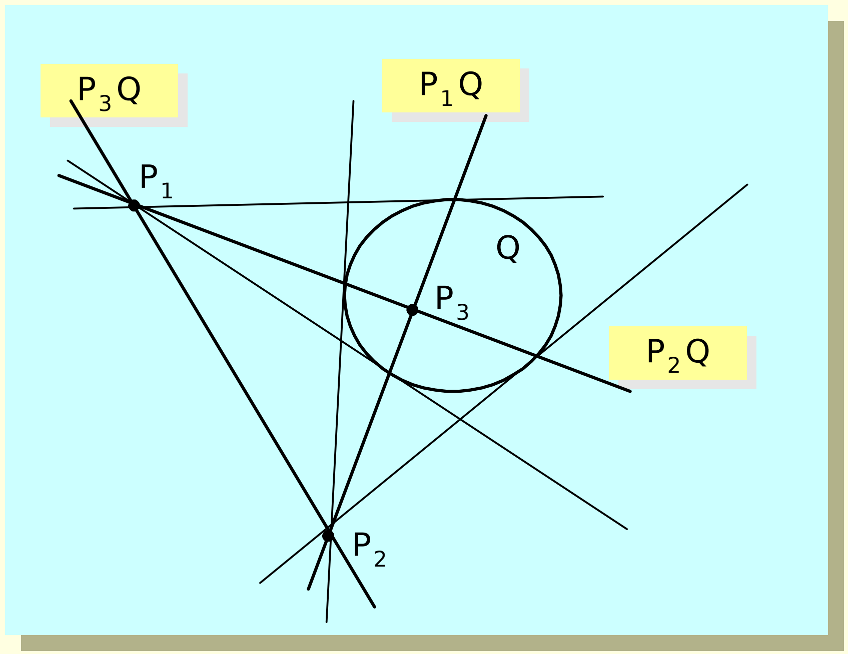
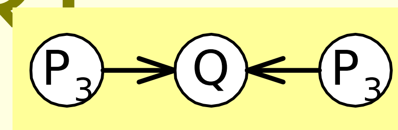


Polar Line To P3





How Does Third Polar Relate to Q?



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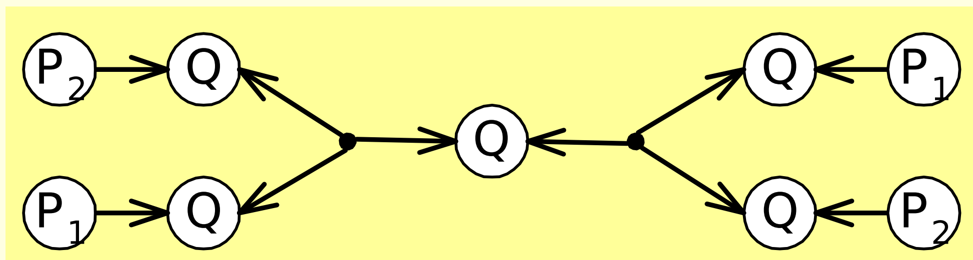
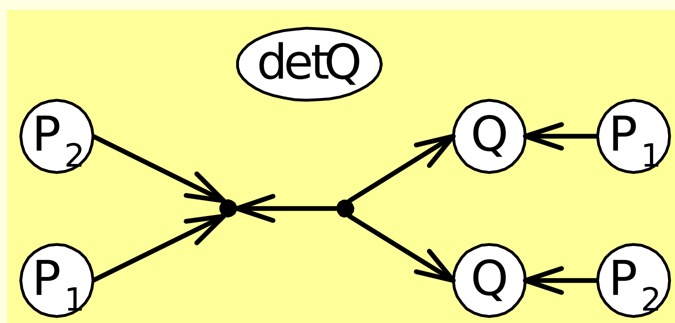
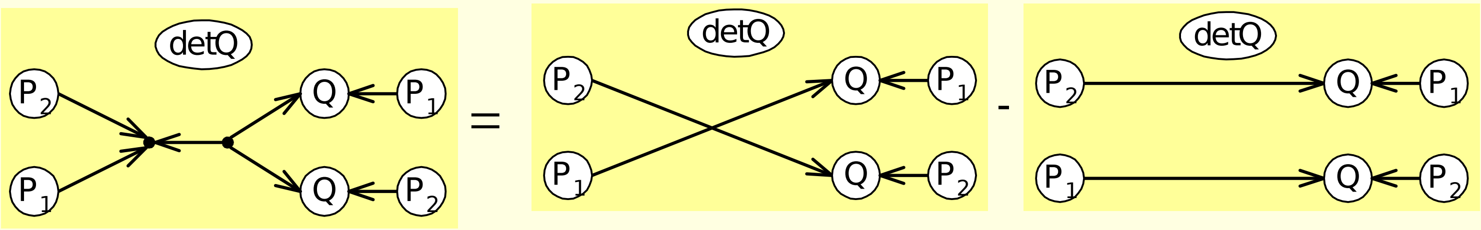
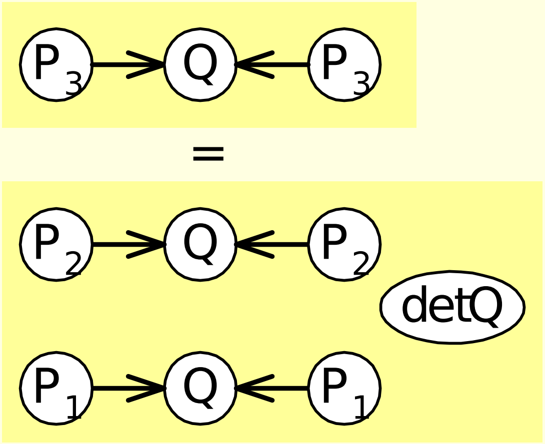
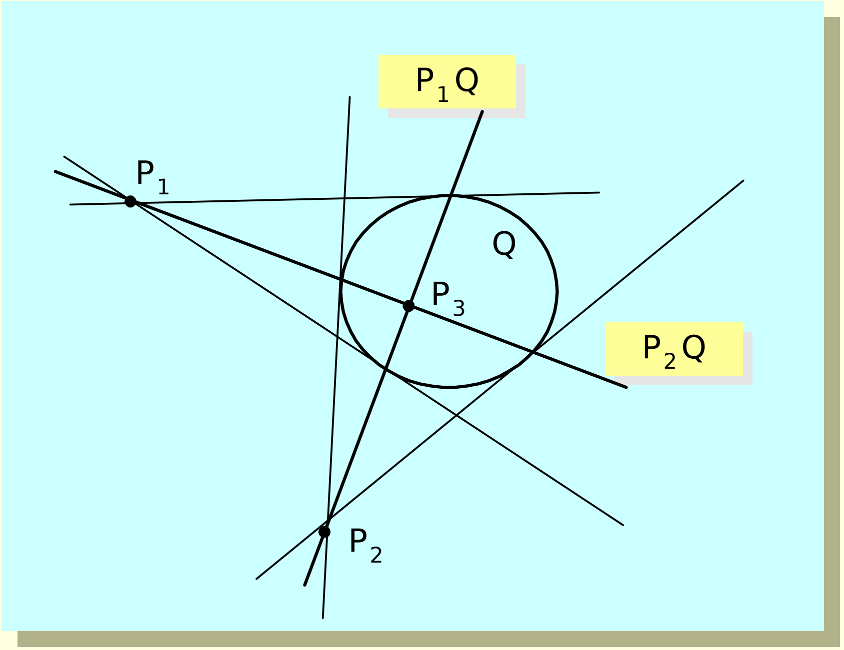




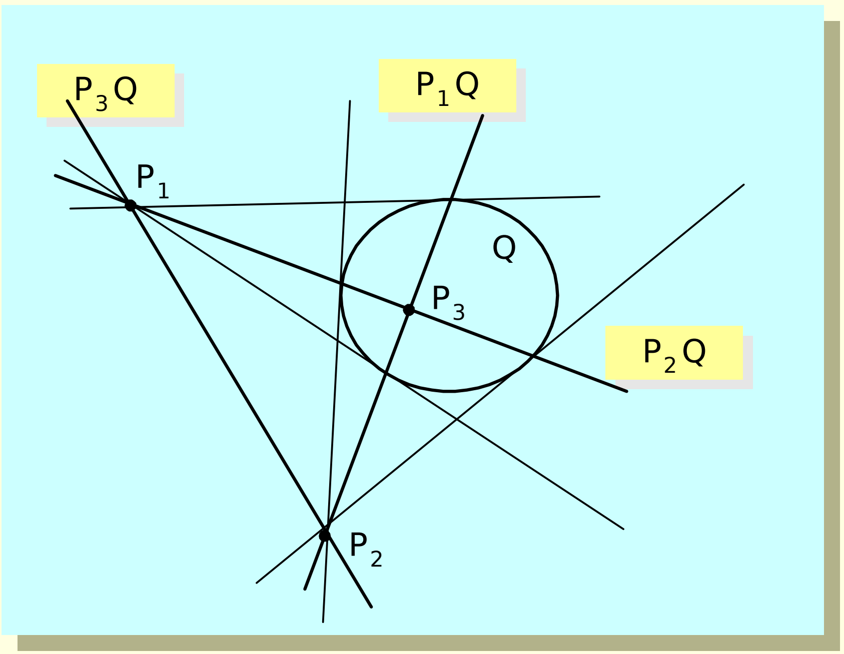
P3 Relation to Q







Make A Transformation Out Of P1,P2,P3

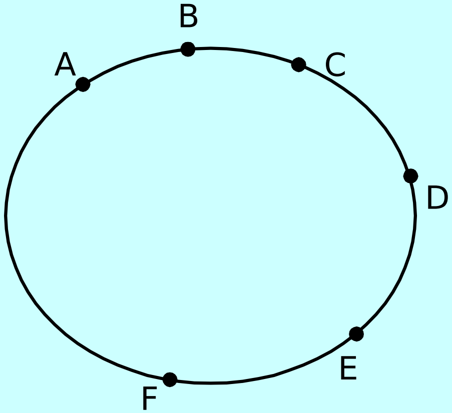


$$\begin{bmatrix} \cdots & \mathbf{P}_1 & \cdots \\ \cdots & \mathbf{P}_2 & \cdots \\ \cdots & \mathbf{P}_3 & \cdots \end{bmatrix} \mathbf{Q} \begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3 \\ \vdots & \vdots & \vdots \end{bmatrix} =$$

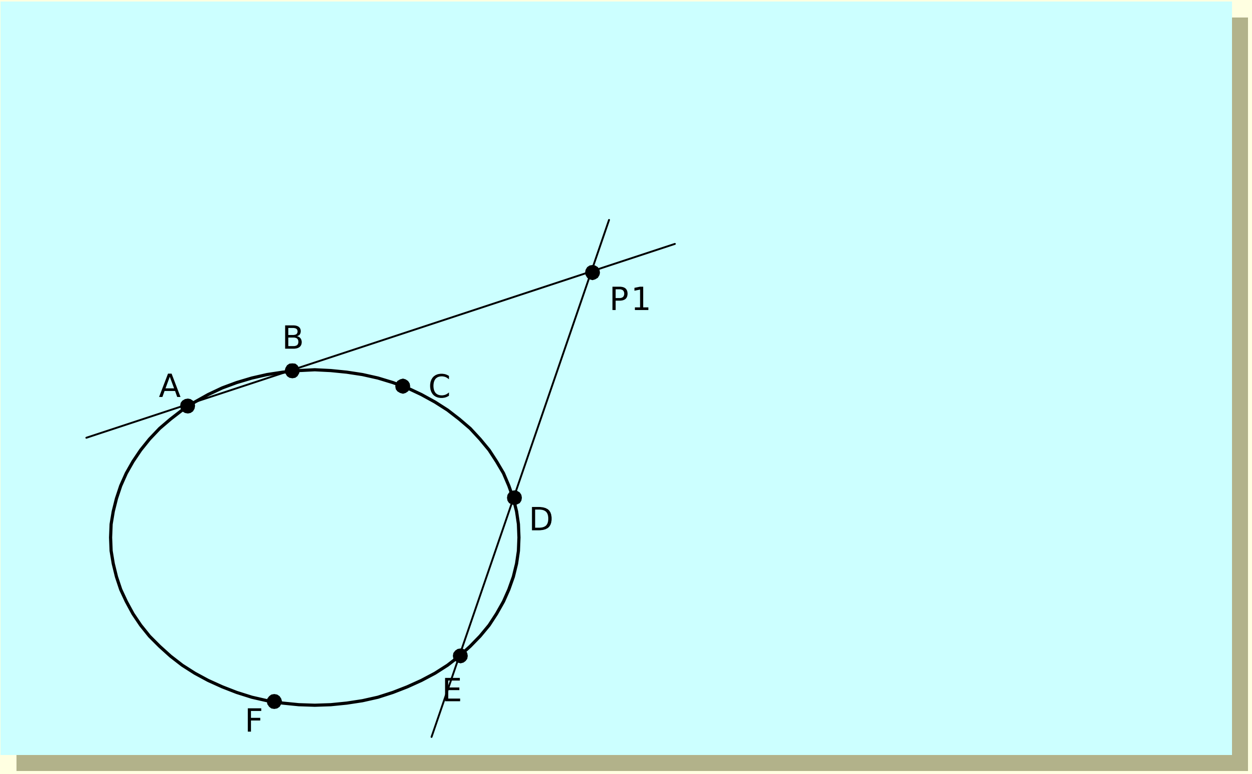
$$\begin{bmatrix} \mathbf{P}_1\mathbf{Q}\mathbf{P}_1^T & \mathbf{P}_1\mathbf{Q}\mathbf{P}_2^T & \mathbf{P}_1\mathbf{Q}\mathbf{P}_3^T \\ \mathbf{P}_2\mathbf{Q}\mathbf{P}_1^T & \mathbf{P}_2\mathbf{Q}\mathbf{P}_2^T & \mathbf{P}_2\mathbf{Q}\mathbf{P}_3^T \\ \mathbf{P}_3\mathbf{Q}\mathbf{P}_1^T & \mathbf{P}_3\mathbf{Q}\mathbf{P}_2^T & \mathbf{P}_3\mathbf{Q}\mathbf{P}_3^T \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{P}_1\mathbf{Q}\mathbf{P}_1^T & 0 & 0 \\ 0 & \mathbf{P}_2\mathbf{Q}\mathbf{P}_2^T & 0 \\ 0 & 0 & \mathbf{P}_3\mathbf{Q}\mathbf{P}_3^T \end{bmatrix}$$

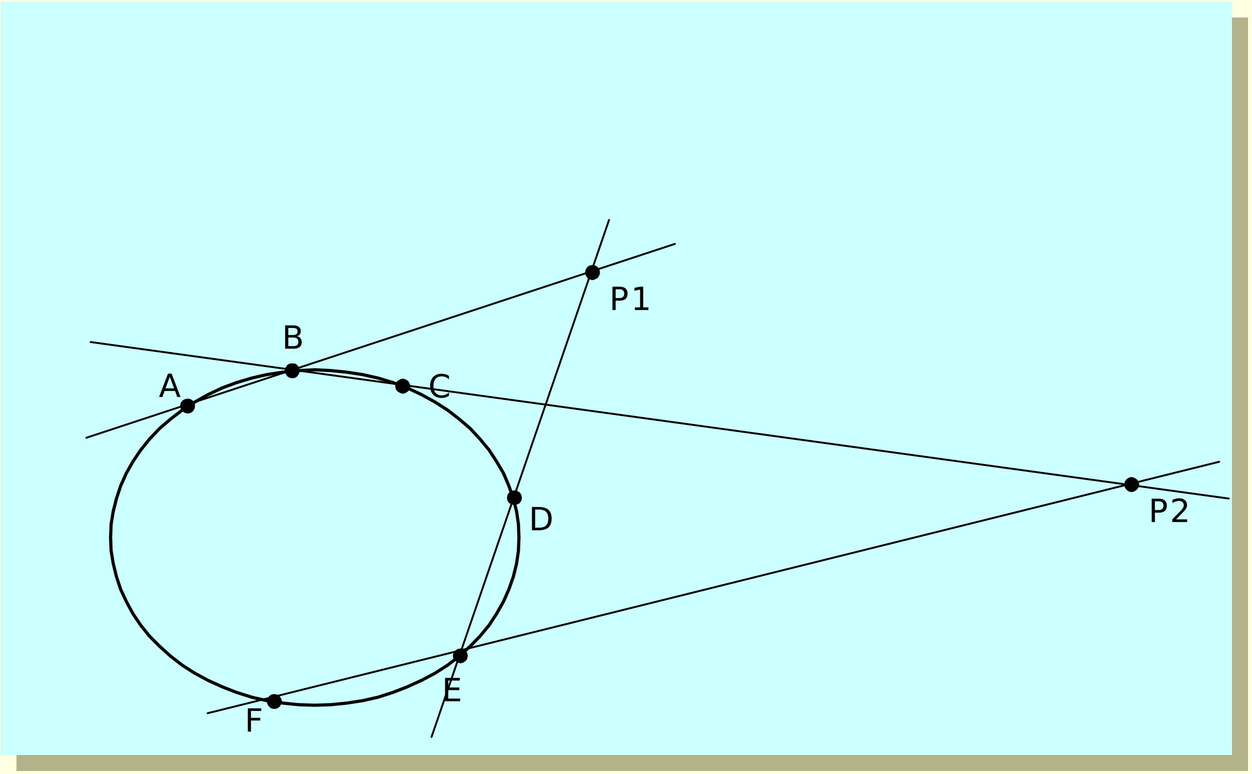
Pascal's (Pappus') Theorem



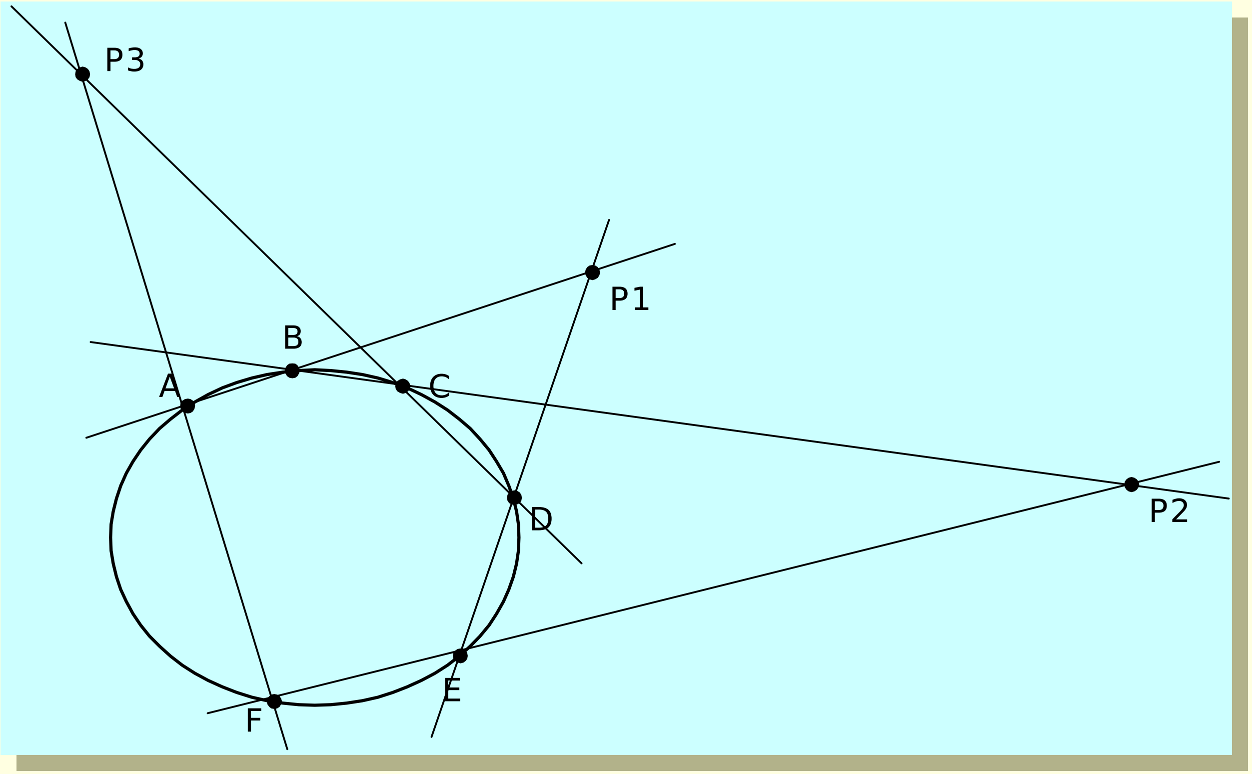
Pascal’s (Pappus’) Theorem



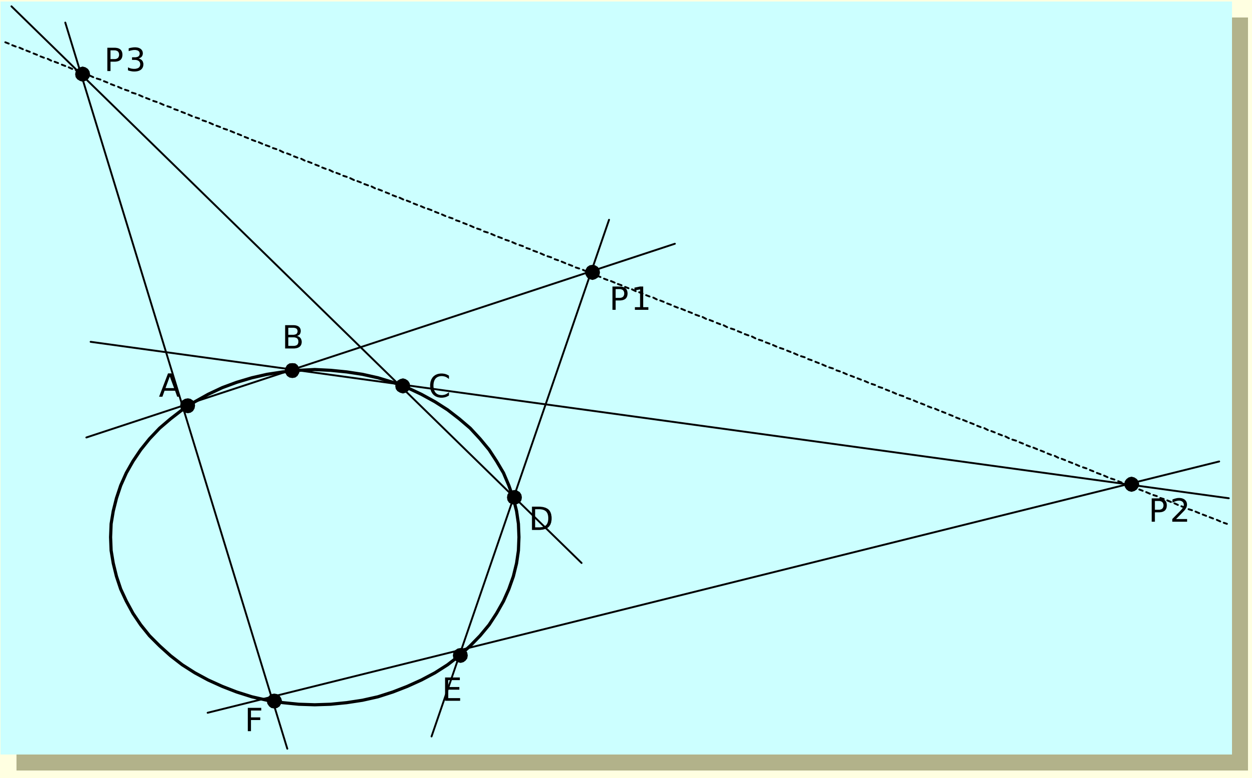
Pascal's (Pappus') Theorem



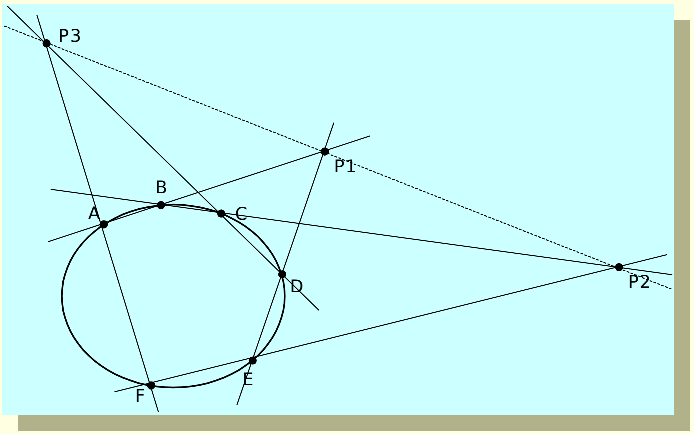
Pascal's (Pappus') Theorem

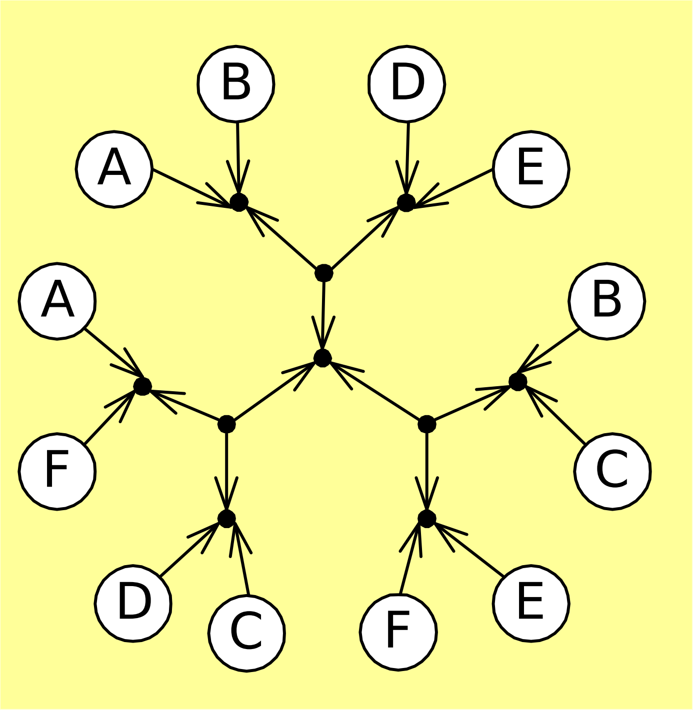


Pascal's (Pappus') Theorem



Pascal's (Pappus') Theorem





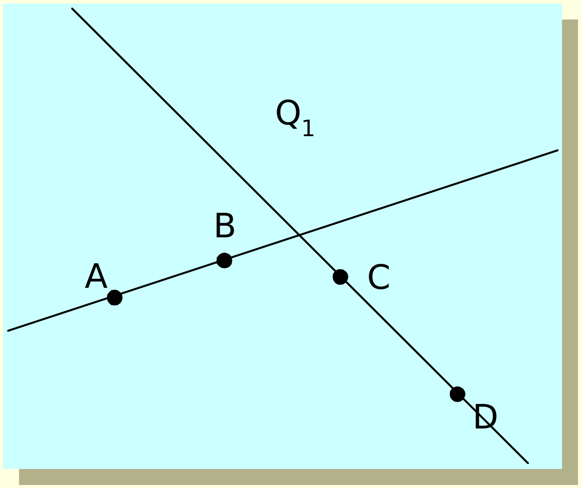
=0

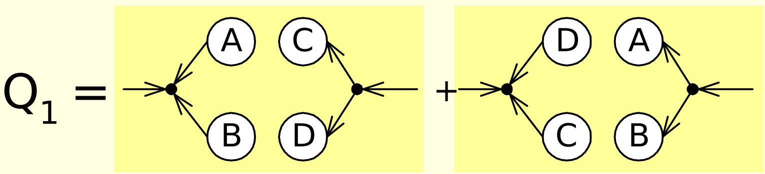
Conic Section on 5 Points

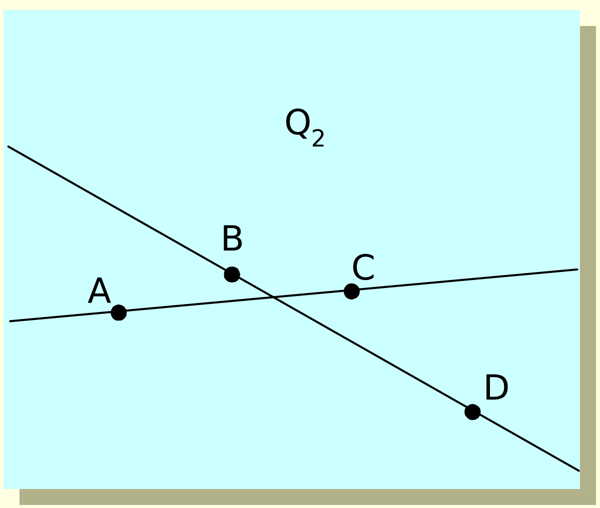
$$Ax^2 + 2Bxy + Cy^2 + 2Dxw + 2Eyw + Fw^2 = 0$$

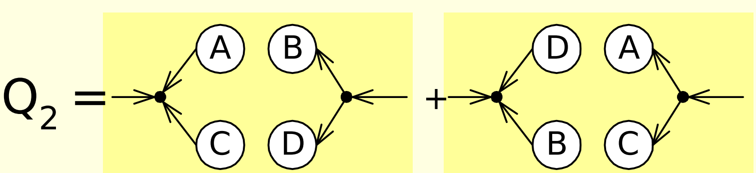
$$\begin{bmatrix} x_1^2 & 2x_1y_1 & y_1^2 & 2x_1w_1 & 2y_1w_1 & w_1^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_5^2 & 2x_5y_5 & y_5^2 & 2x_5w_5 & 2y_5w_5 & w_5^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A (Better) Way

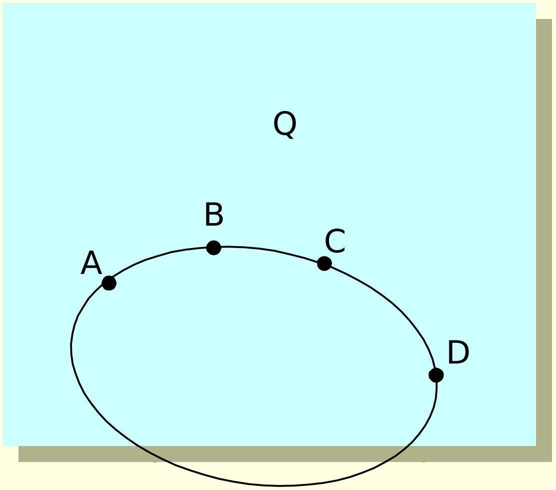






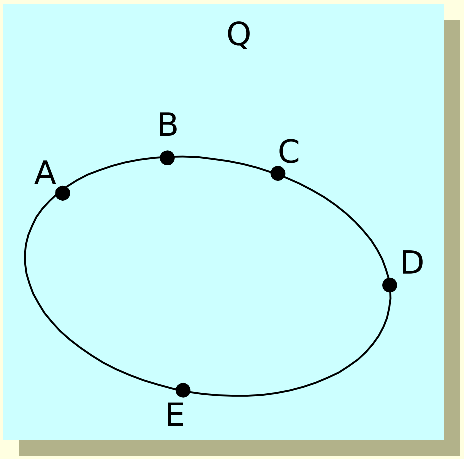


Linear Combo of Q1 and Q2



$$\mathbf{Q} = a\,\mathbf{Q}_1 + b\,\mathbf{Q}_2$$

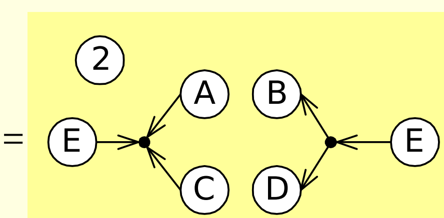
Pick α,β to make Point E be on Q

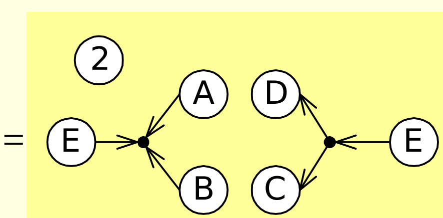


$$\begin{aligned} 0 &= \mathbf{E}\mathbf{Q}\mathbf{E}^T \\ &= \mathbf{E}\left(a\mathbf{Q}_1 + b\mathbf{Q}_2\right)\mathbf{E}^T \\ &= a\left(\mathbf{E}\mathbf{Q}_1\mathbf{E}^T\right) + b\left(\mathbf{E}\mathbf{Q}_2\mathbf{E}^T\right) \end{aligned}$$

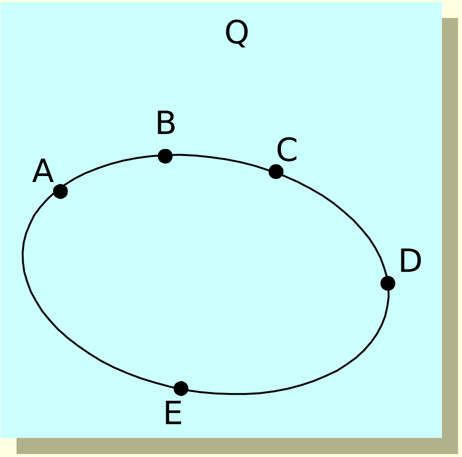
$$a = \mathbf{E}\mathbf{Q}_2\mathbf{E}^T$$

$$b = -\mathbf{E}\mathbf{Q}_1\mathbf{E}^T$$

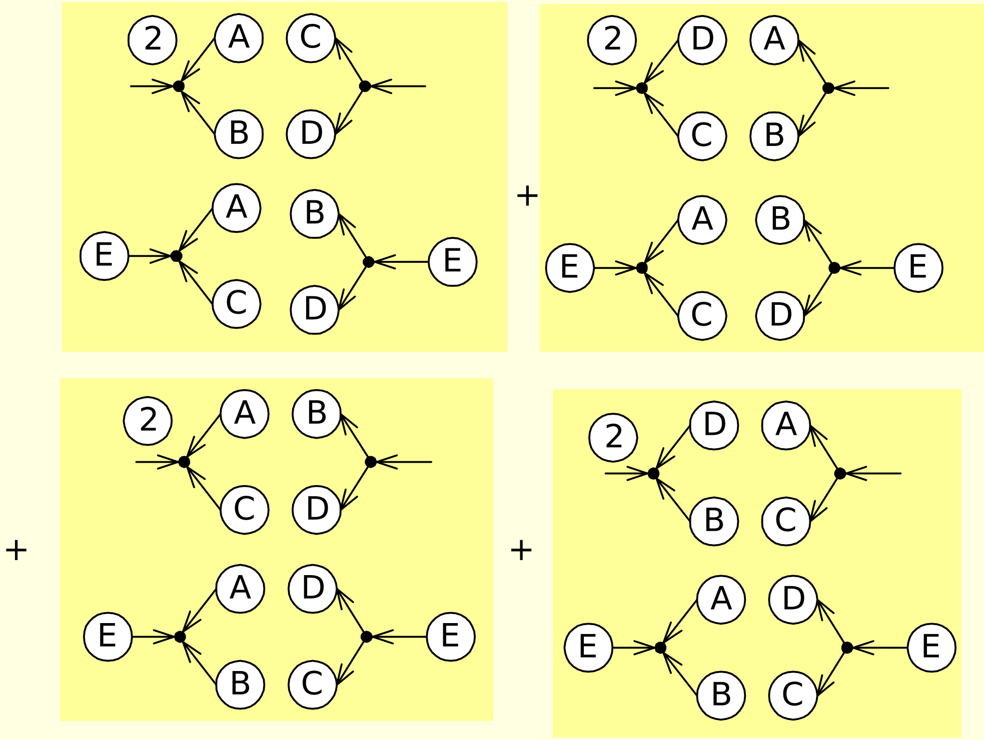
$\alpha =$ 

$\beta =$ 

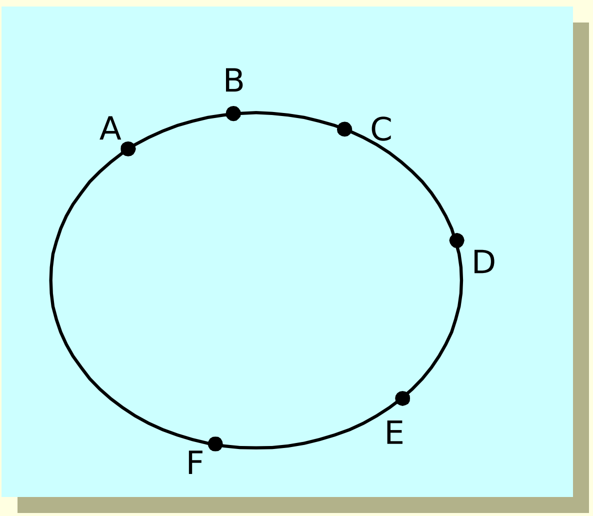
Quadratic on 5 Points

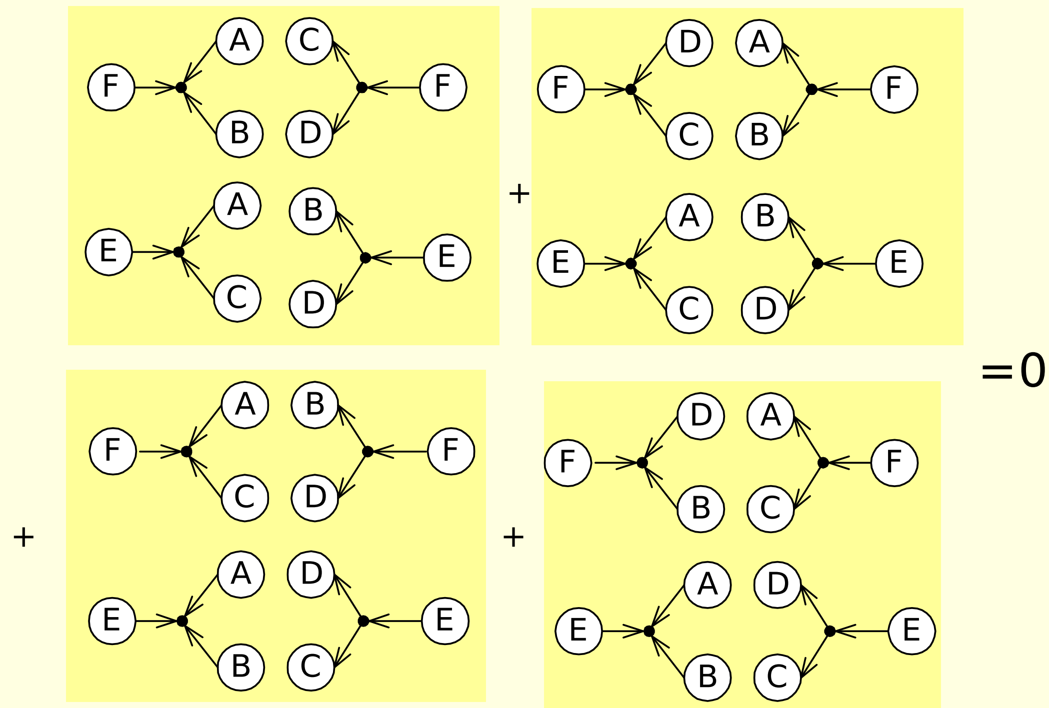


$$\mathbf{Q} = a\,\mathbf{Q}_1 + b\,\mathbf{Q}_2$$

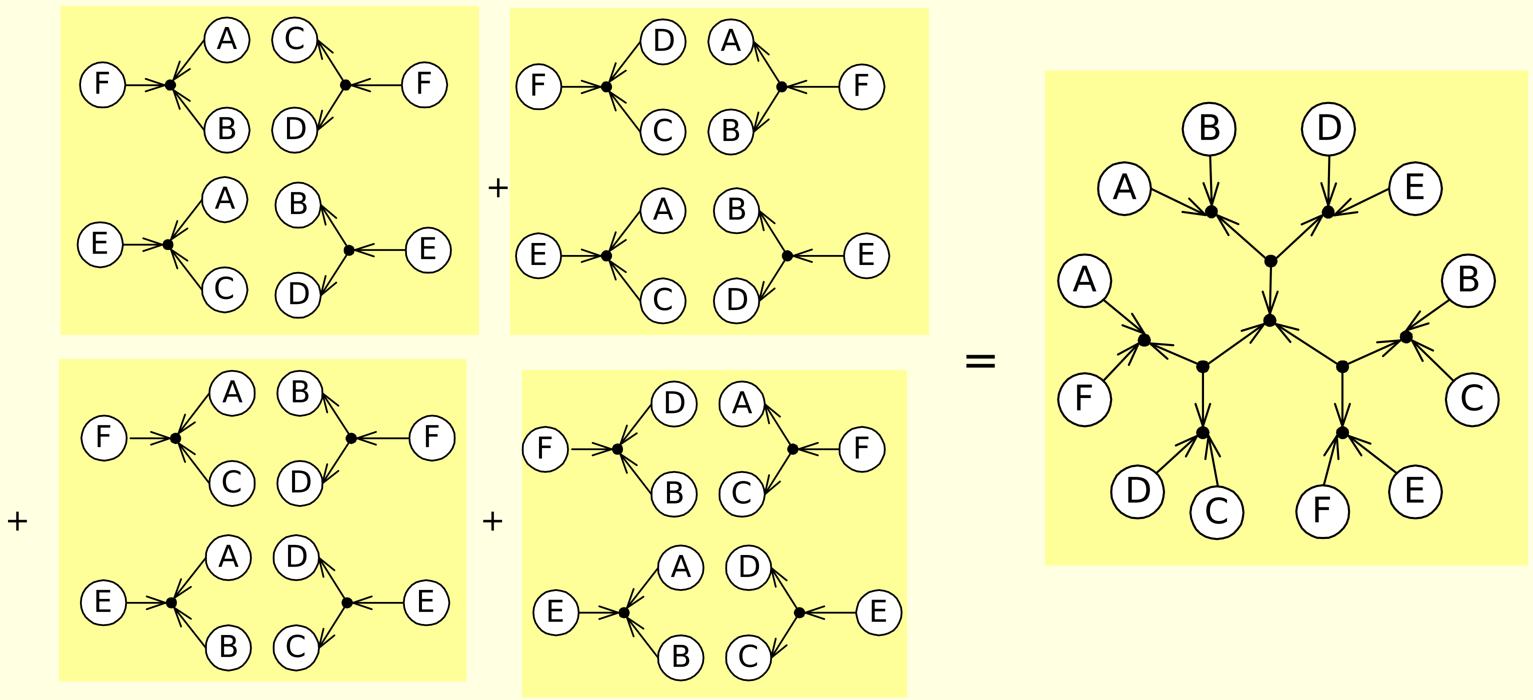


Six Points (ABCDEF) on Quadratic





Pascal’s Theorem



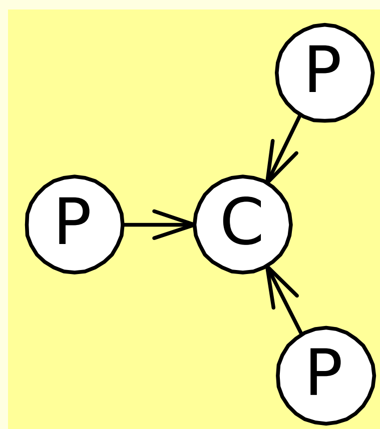
2DH Curves

Cubic

The Cubic Curve Equation

$$\begin{aligned} &Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3 \\ &+ 3Ex^2w + 6Fxyw + 3Gy^2w \\ &+ 3Hxw^2 + 3Jyw^2 \\ &+ Kw^3 = 0 \end{aligned}$$

$$\left\{ \begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} \begin{bmatrix} A & B & E \\ B & C & F \\ E & F & H \end{bmatrix} & \begin{bmatrix} B & C & F \\ C & D & G \\ F & G & J \end{bmatrix} & \begin{bmatrix} E & F & H \\ F & G & J \\ H & J & K \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$



$= 0$

Standard Positions

$$\begin{aligned}&Ax^3+3Bx^2y+3Cxy^2+Dy^3\\&+3Ex^2w+6Fxyw+3Gy^2w\\&+3Hxw^2+3Jyw^2\\&+Kw^3\\&=0\end{aligned}$$

Transform to make some coefficients zero

$$\begin{aligned}&Ax^3+3Bx^2y+3Cxy^2+Dy^3\\&+3Ex^2w+6Fxyw+3Gy^2w\\&+3Hxw^2+3Jyw^2\\&+Kw^3\\&=0\end{aligned}$$

$$\begin{aligned}&Ax^3+3Bx^2y+3Cxy^2+Dy^3\\&+3Ex^2w+6Fxyw+3Gy^2w\\&+3Hxw^2+3Jyw^2\\&+Kw^3\\&=0\end{aligned}$$

$$\begin{aligned}&Ax^3+3Bx^2y+3Cxy^2+Dy^3\\&+3Ex^2w+6Fxyw+3Gy^2w\\&+3Hxw^2+3Jyw^2\\&+Kw^3\\&=0\end{aligned}$$

My Favorite Standard Position

$$Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3$$

$$+3Ex^2w + 6Fxyw + 3Gy^2w$$

$$+3Hxw^2 + 3Jyw^2$$

$$+Kw^3$$

$$=0$$

$$-3Gy^2w = Ax^3 + 3Ex^2w + 3Hxw^2 + Kw^3$$

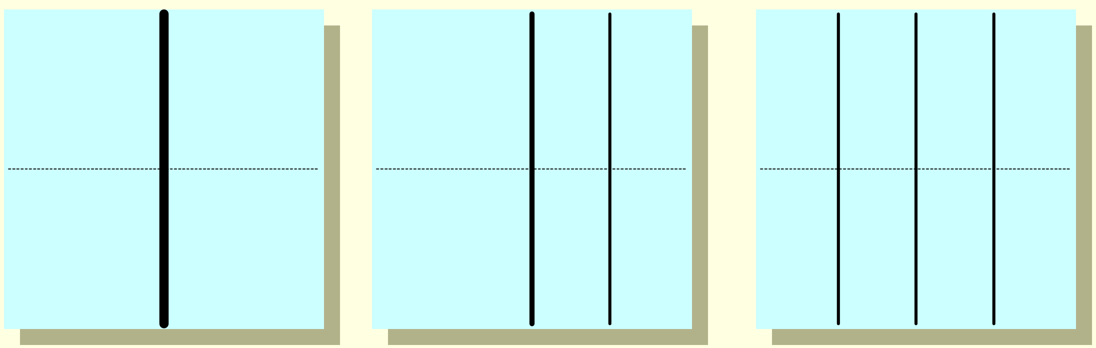
The Catalog – Reducible Cubics

$$-3Gy^2w = Ax^3 + 3Ex^2w + 3Hxw^2 + Kw^3$$

$$G = 0$$

ß

$$0 = Ax^3 + 3Ex^2w + 3Hxw^2 + Kw^3$$



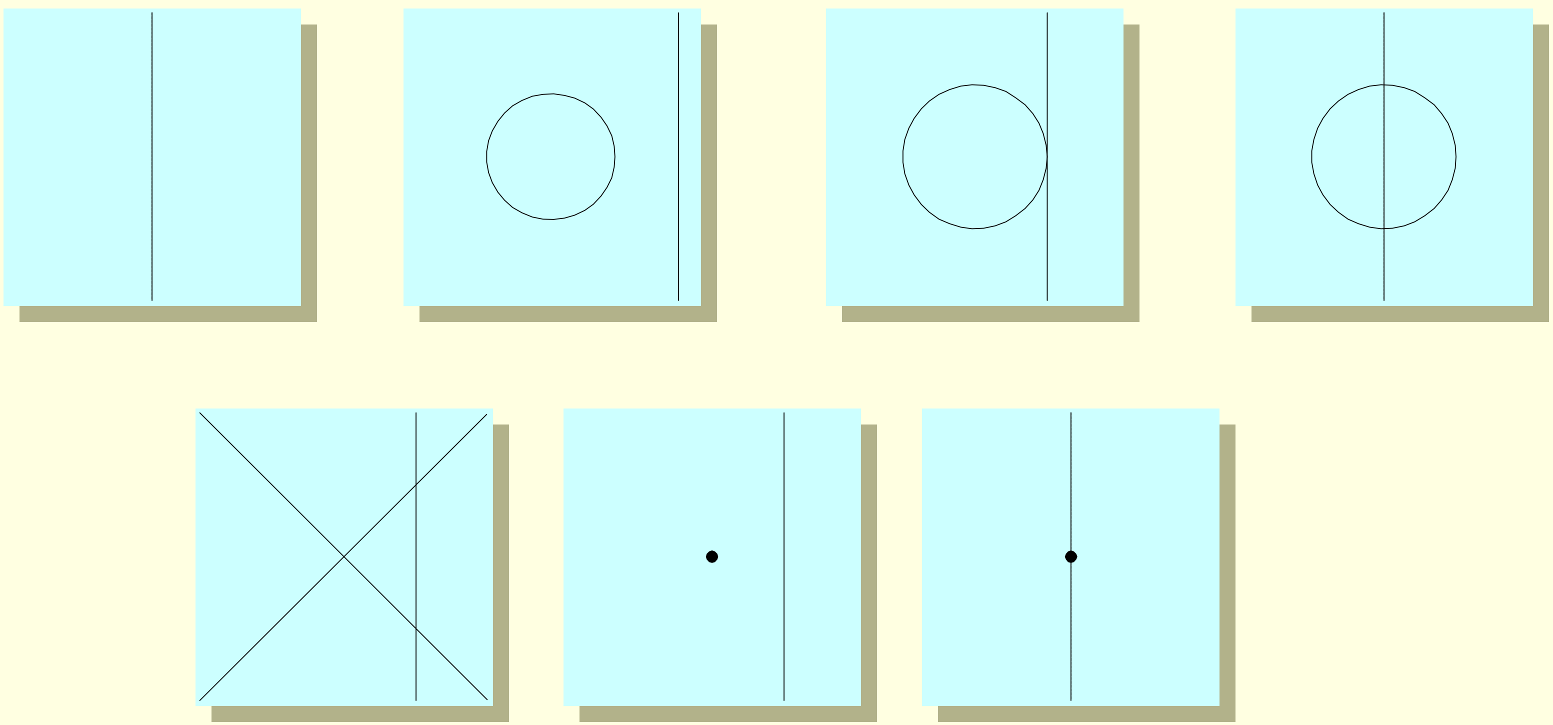
The Catalog – Reducible Cubics

$$-3Gy^2w = Ax^3 + 3Ex^2w + 3Hxw^2 + Kw^3$$

$$A = 0$$

ß

$$0 = \left(3Ex^2 + 3Hxw + Kw^2 - 3Gy^2\right)w$$



The Catalog

$G \neq 0$

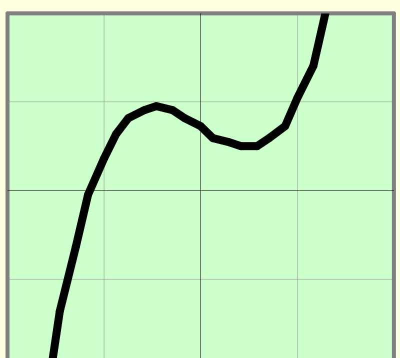
$A \neq 0$

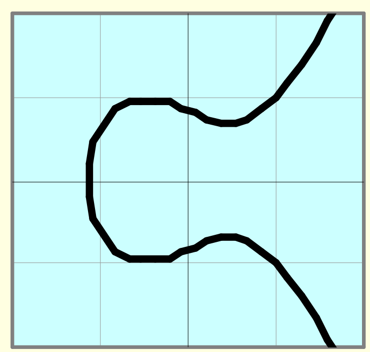
$$-3Gy^2w = Ax^3 + 3Ex^2w + 3Hxw^2 + Kw^3$$

$$\Downarrow$$

$$y^2w = x^3 + 3Hxw^2 + Kw^3$$

$$Y = \sqrt{X^3 + cX + d}$$





Not A Two Parameter Class

$$Y = \sqrt{X^3 + cX + d}$$

Scale in X
and Y

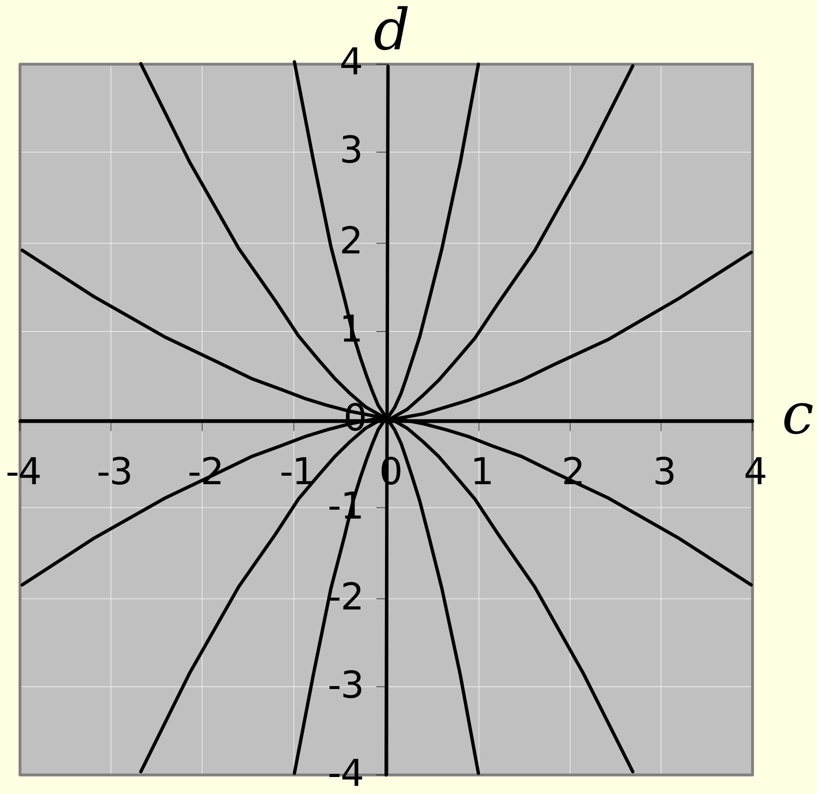
$$sY = \sqrt{\left(\sqrt[3]{s^2}X\right)^3 + c\left(\sqrt[3]{s^2}X\right) + d}$$

$$Y = \sqrt{X^3 + \hat{c}X + \hat{d}}$$

$$\hat{c} = cs^{-4/3}, \hat{d} = ds^{-2}$$

$$\frac{c^3}{d^2} = \frac{\hat{c}^3}{\hat{d}^2} = \text{constant}$$

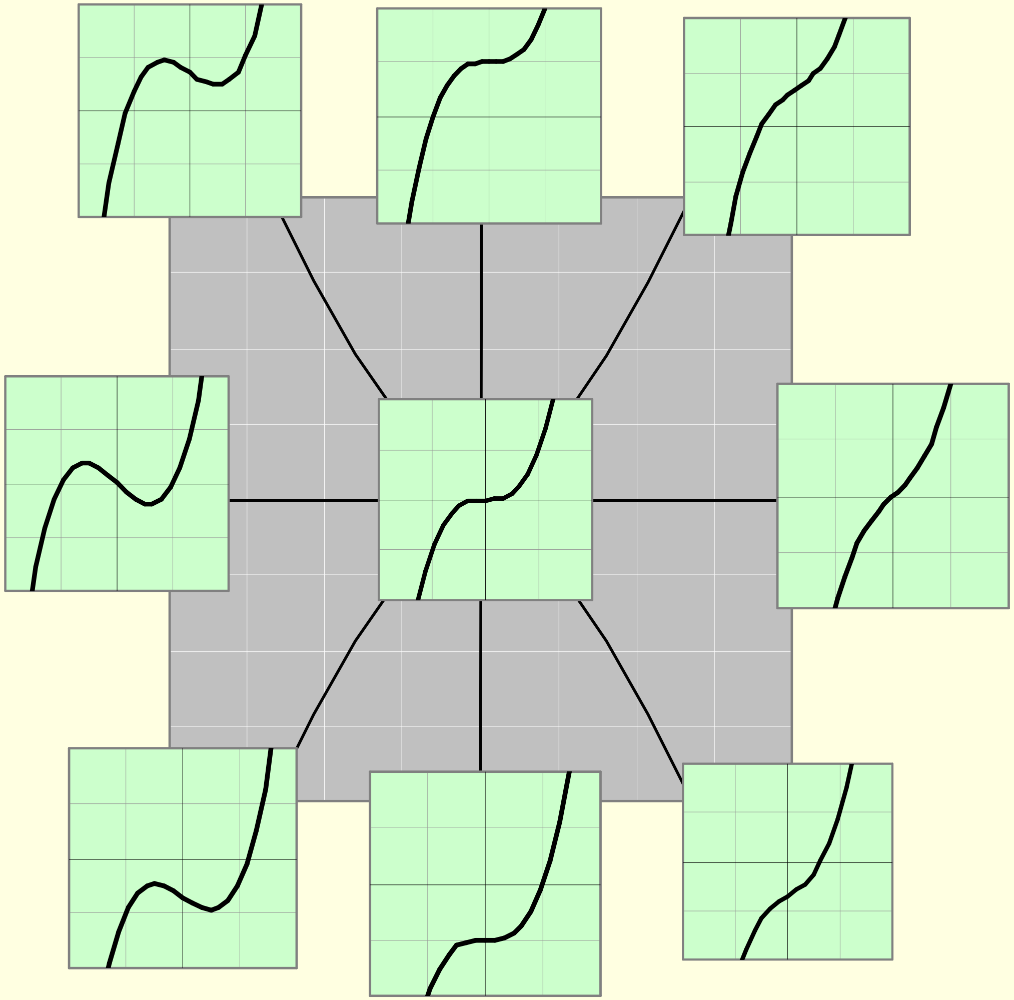
Space of Irreducible Cubics



$$\frac{c^3}{d^2} = \text{constant}$$

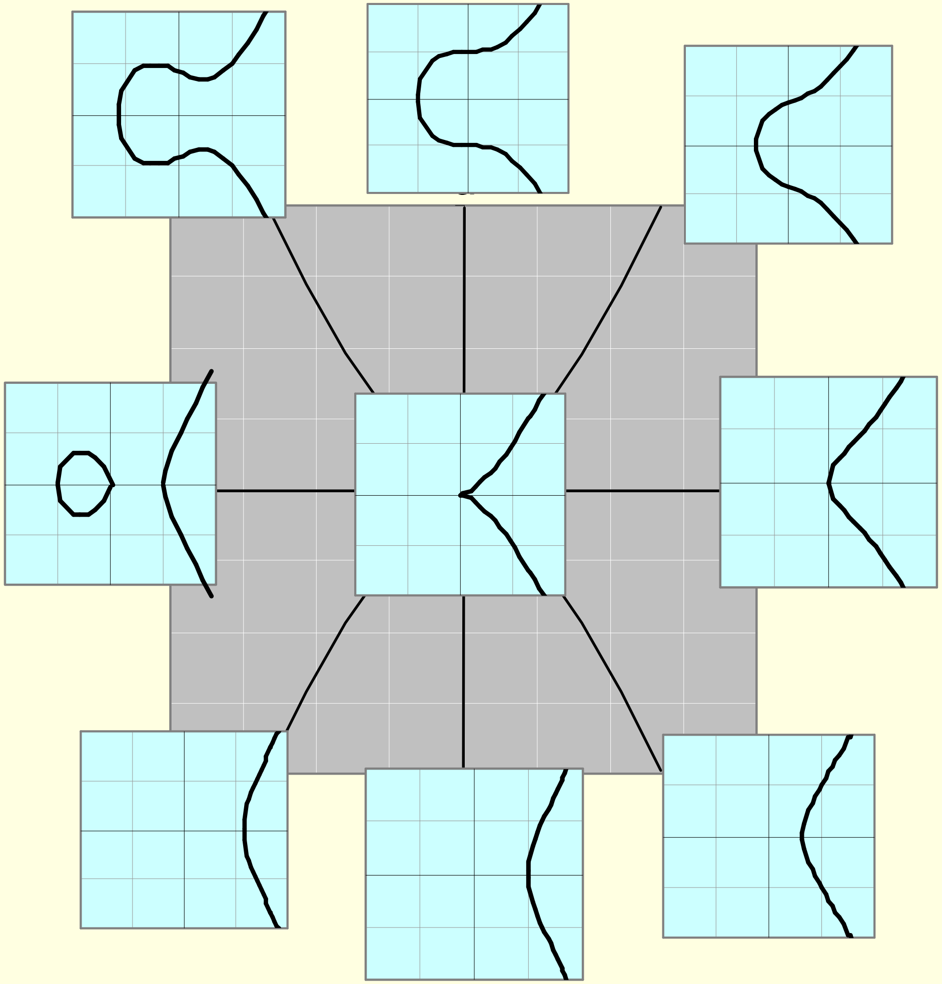
Plot Y squared

$$Y^2 = X^3 + cX + d$$



Samples of Irreducible Cubics

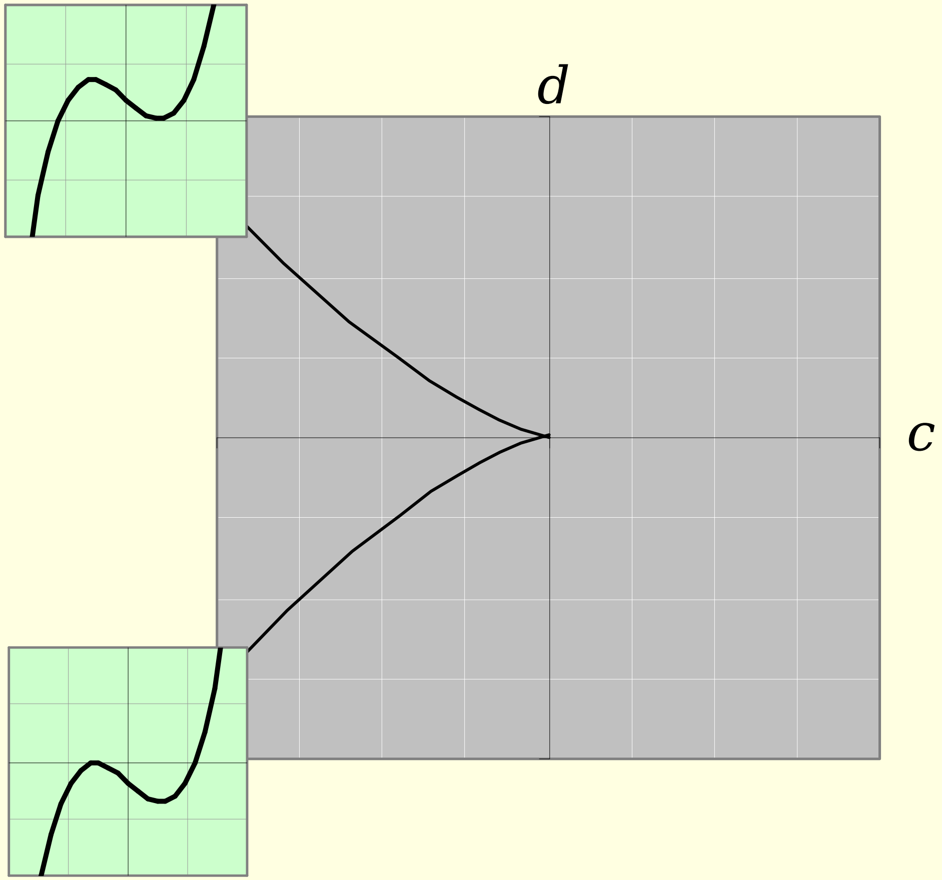
$$Y = \sqrt{X^3 + cX + d}$$



Particularly Interesting Cases

$$Y^2 = X^3 - 3X + 2$$

$$= (X+2)(X-1)^2$$

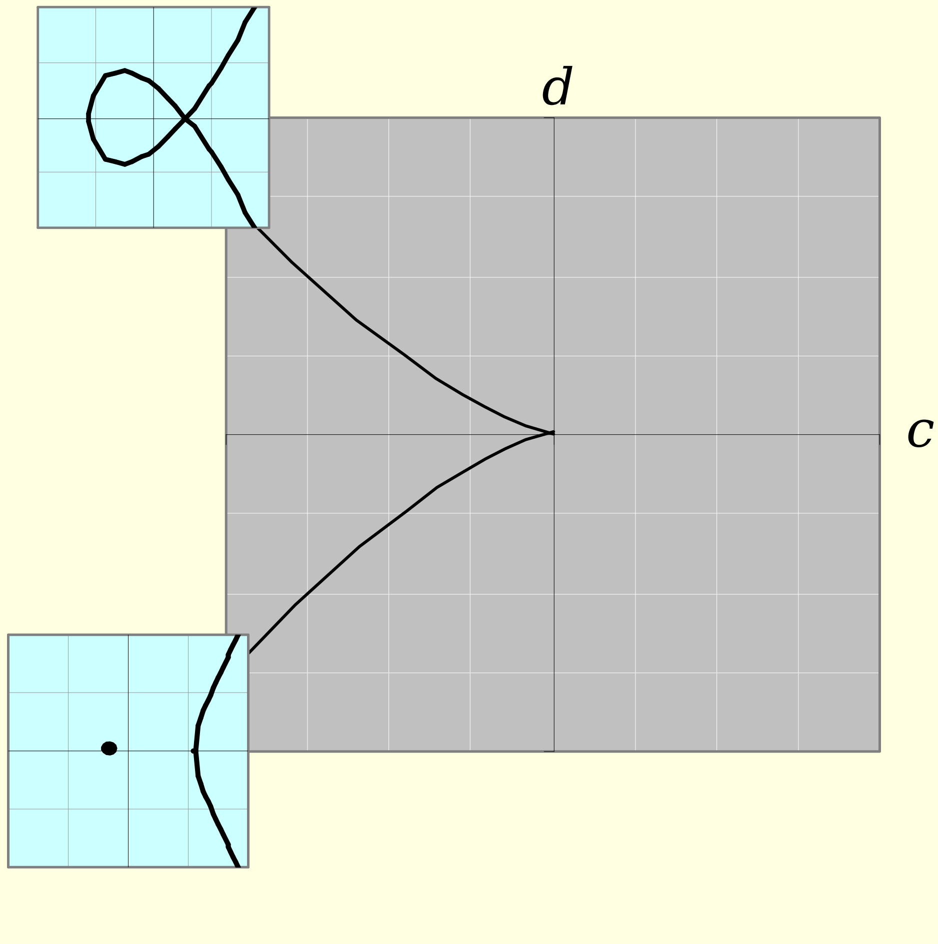


$$Y^2 = X^3 - 3X - 2$$

$$= (X-2)(X+1)^2$$

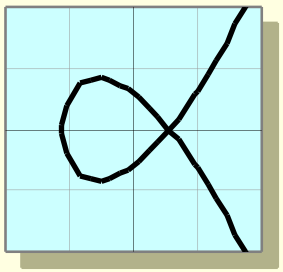
Acnode and Crunode

$$Y^2 = X^3 - 3X + 2$$
$$= (X+2)(X-1)^2$$

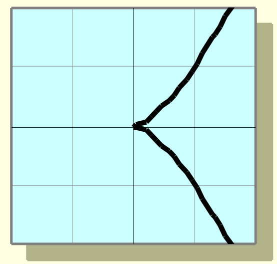


$$Y^2 = X^3 - 3X - 2$$
$$= (X-2)(X+1)^2$$

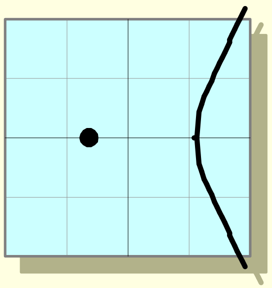
Irreducible Cubic Curves



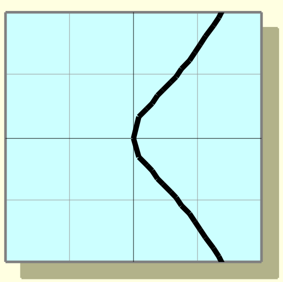
$$x^3 - 3xw^2 + 2w^3 - y^2w = 0$$

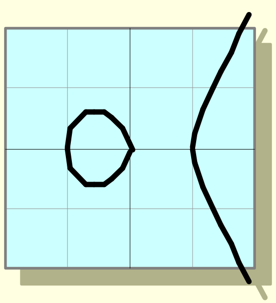


$$0 = x^3 - y^2w$$



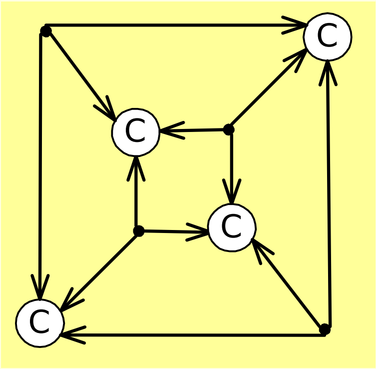
$$x^3 - 3xw^2 - 2w^3 - y^2w = 0$$



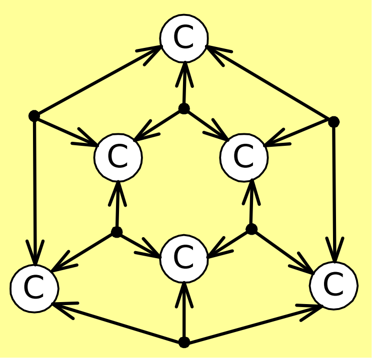


$$y^2w = x^3 + cxw^2 + dw^3$$

Invariants



$$I_{cube} = 24G^2\left(E^2 - AH\right)$$



$$I_{hexagon} = 24G^3\left(A\left(EH - AK\right) + 2E\left(AH - E^2\right)\right)$$

$$\mathbf{D} = 16A^3G^6\left(A^3K^2 + 4H^3\right)$$

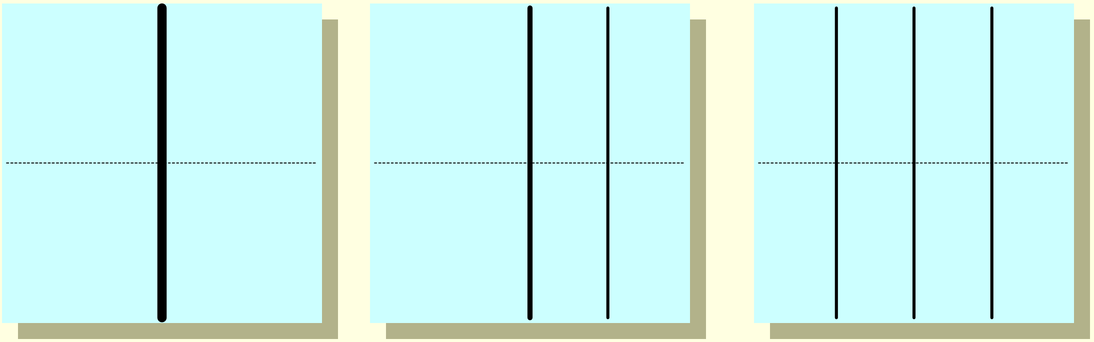
Doubly Reducible

$$G = 0$$

$$I_{cube} = 0$$

$$I_{hexagon} = 0$$

$$\mathbf{D} = 0$$

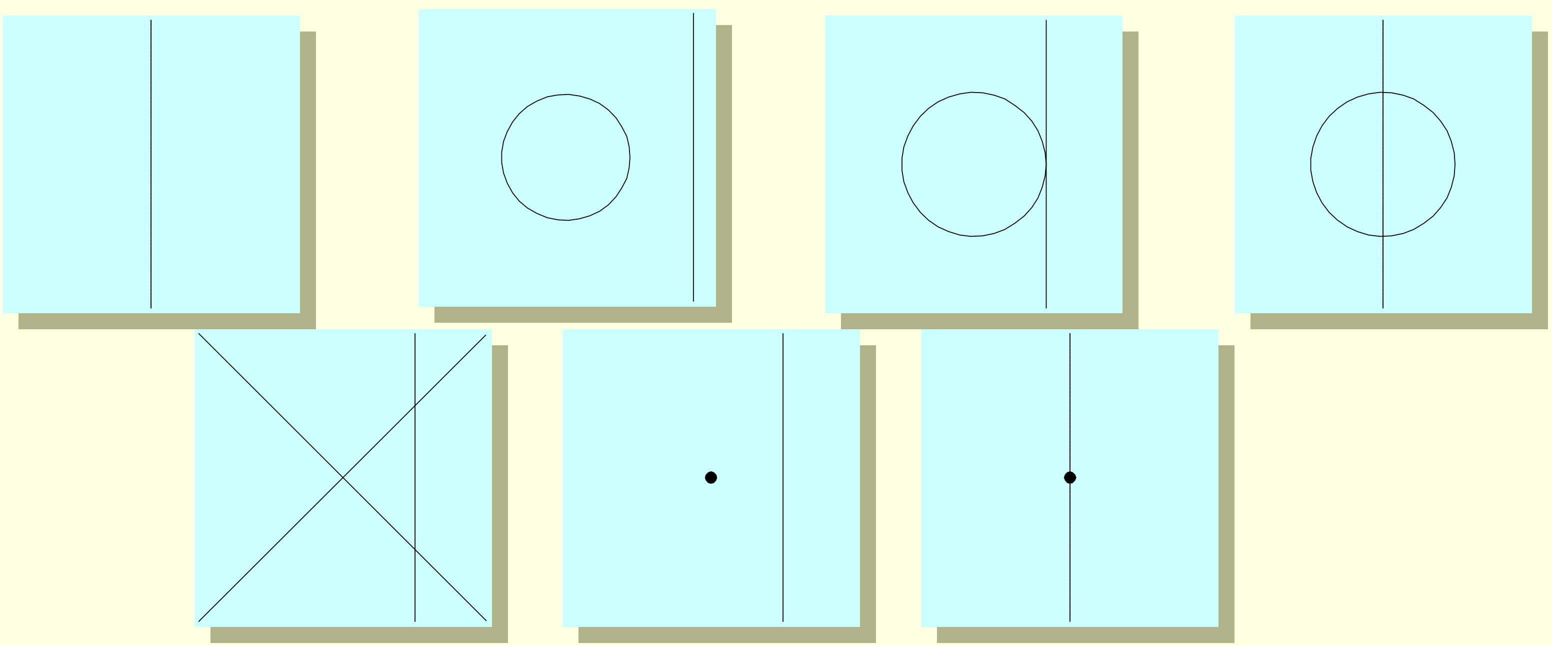


Reducible

$$I_{cube} = 24G^2E^2$$

$$A = 0 \qquad I_{hexagon} = -48G^3E^3$$

$$\mathbf{D} = 0$$

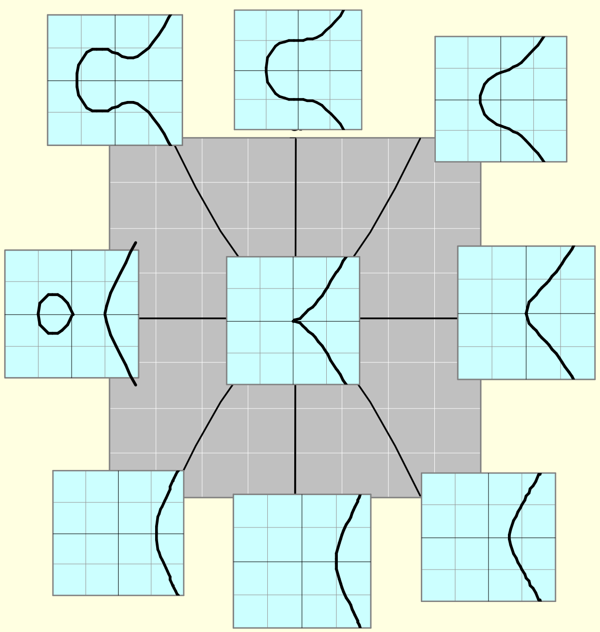


Irreducible

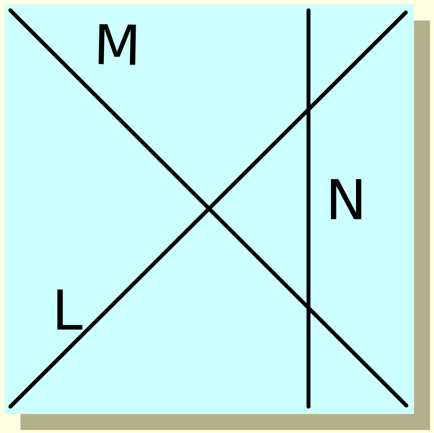
$$I_{cube} = -24H$$

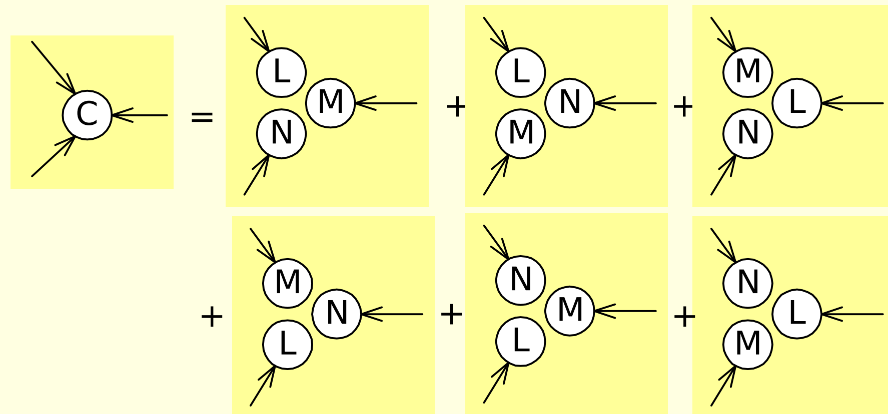
$$G = 1, A = 1, E = 0 \Rightarrow \quad I_{hexagon} = -24K$$

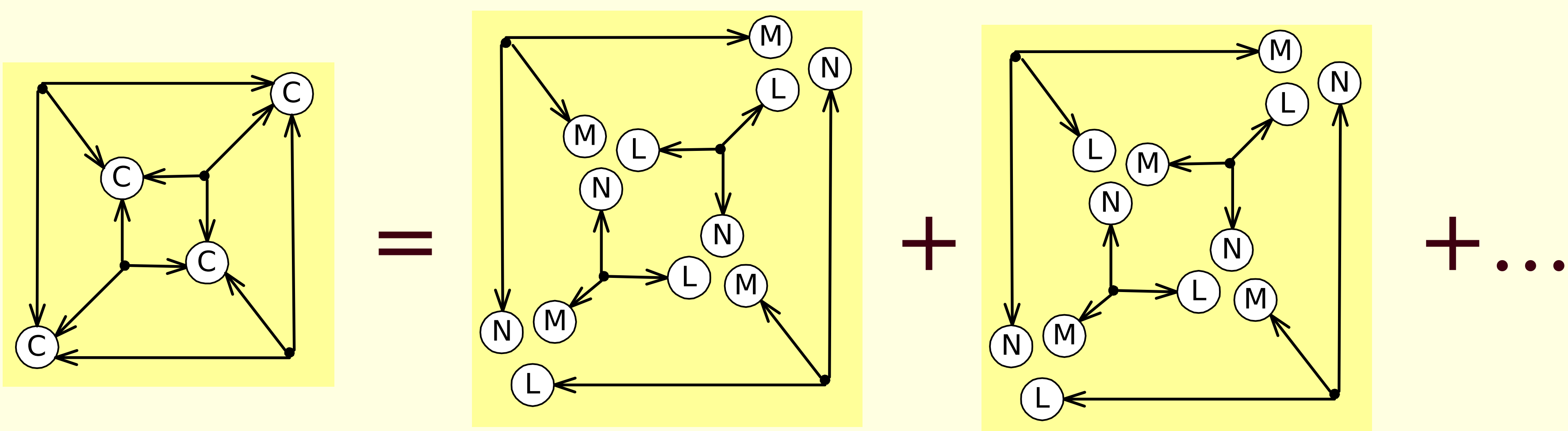
$$\Delta = 16\left(K^2 + 4H^3\right)$$



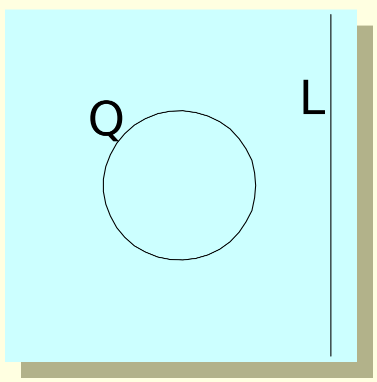
SubAtomic Cubics - Reducible

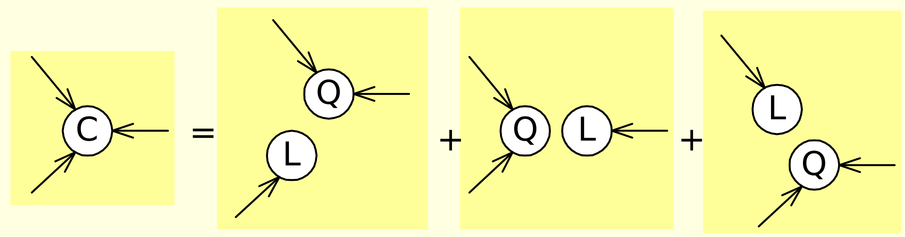


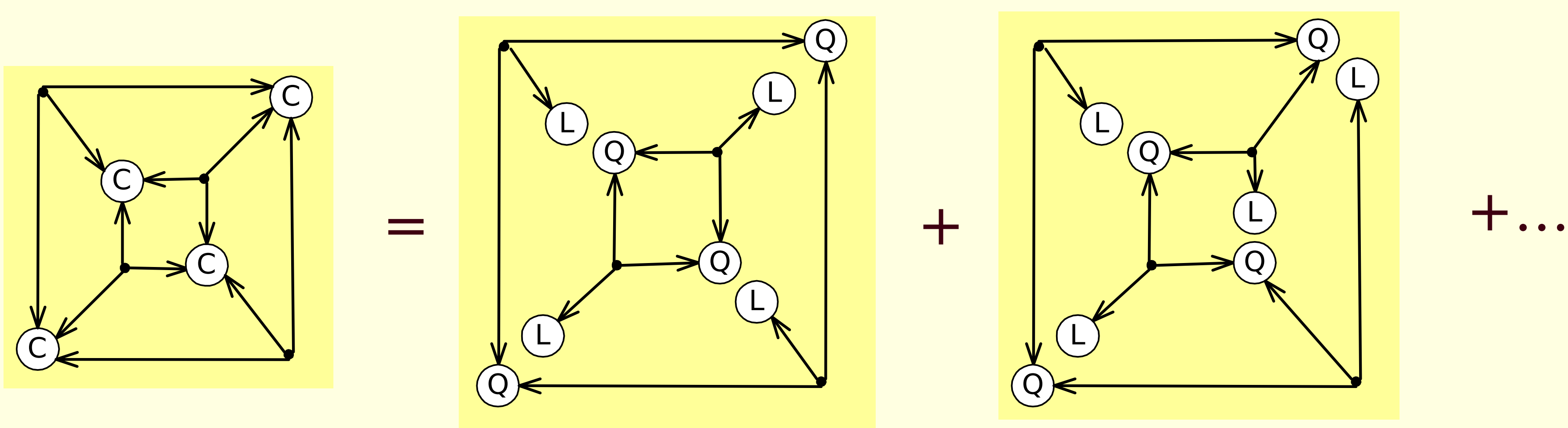




SubAtomic Cubics - Reducible

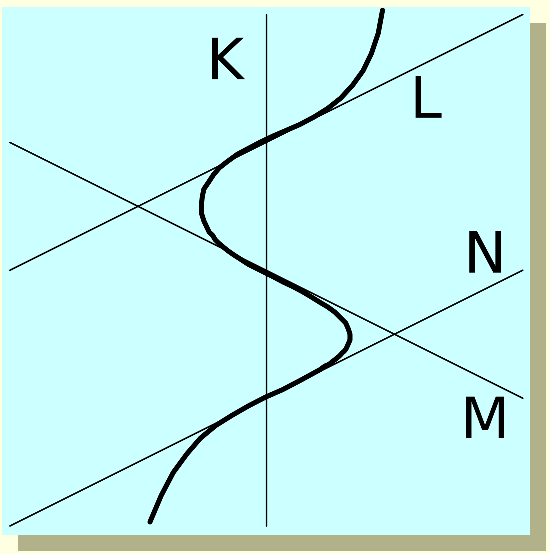


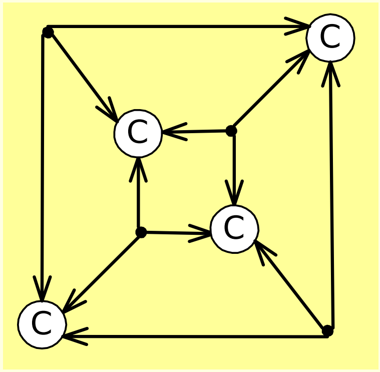


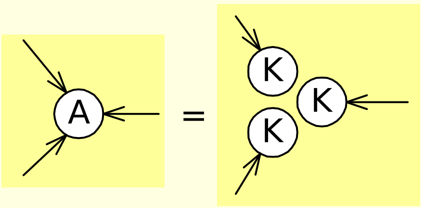


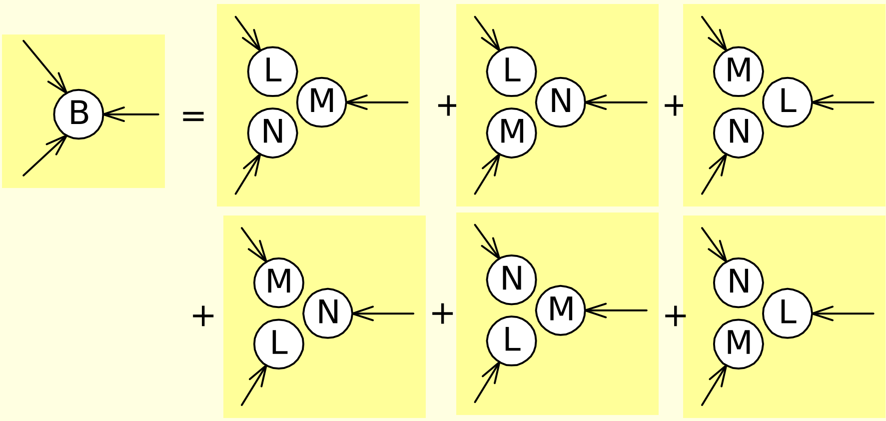
SubAtomic Cubics - Irreducible

$$\mathbf{C} = a\mathbf{A} + b\mathbf{B}$$



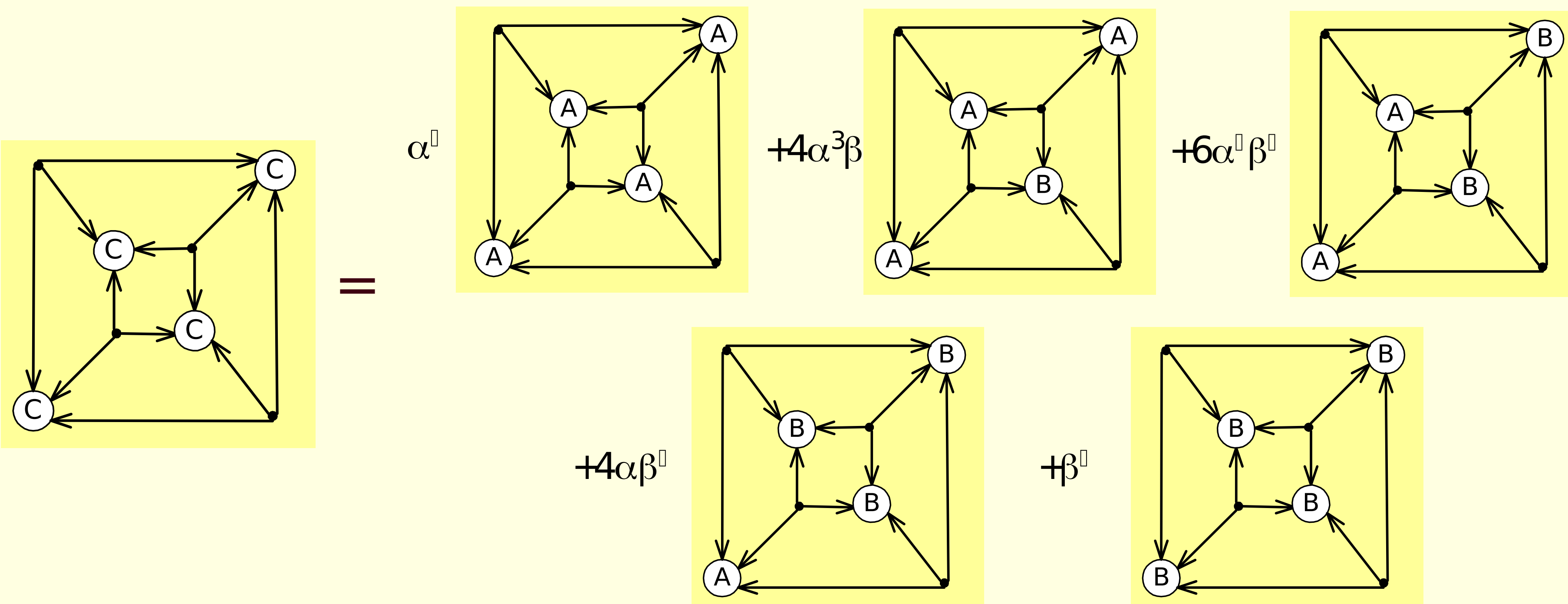




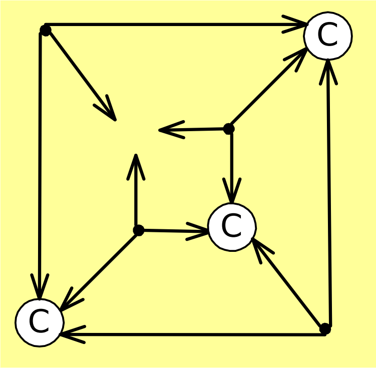


SubAtomic Cubics - Irreducible

$$\mathbf{C} = a\,\mathbf{A} + b\,\mathbf{B}$$



Caylean



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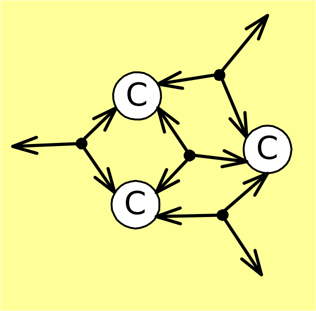
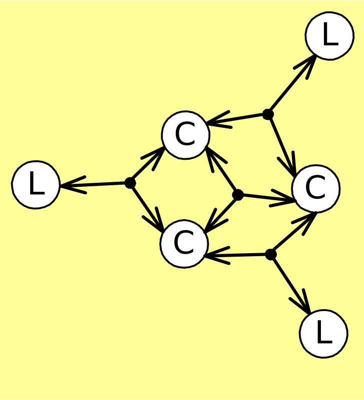




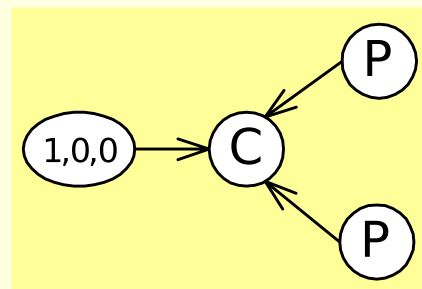
= 0 Means what?

First Derivatives

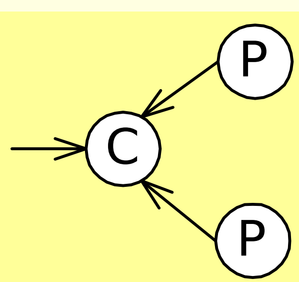
$$f(x,y,w) = Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3$$
$$+ 3Ex^2w + 6Fxyw + 3Gy^2w$$
$$+ 3Hxw^2 + 3Jyw^2$$
$$+ Kw^3$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \left\{ \begin{bmatrix} A & B & E \\ B & C & F \\ E & F & H \end{bmatrix} \begin{bmatrix} B & C & F \\ C & D & G \\ F & G & J \end{bmatrix} \begin{bmatrix} E & F & H \\ F & G & J \\ H & J & K \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ w \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

$$\frac{\partial f}{\partial x} = f_x = 3Ax^2 + 6Bxy + 3Cy^2$$
$$+ 6Exw + 6Fyw$$
$$+ 3Hw^2$$



$$\begin{bmatrix} f_x \\ f_y \\ f_w \end{bmatrix} = 3$$



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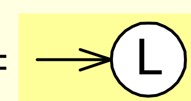


Second Derivatives

$$\begin{bmatrix} x & y & w \end{bmatrix} \left(\begin{bmatrix} A & B & E \\ B & C & F \\ E & F & H \end{bmatrix} x + \begin{bmatrix} B & C & F \\ C & D & G \\ F & G & J \end{bmatrix} y + \begin{bmatrix} E & F & H \\ F & G & J \\ H & J & K \end{bmatrix} w \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

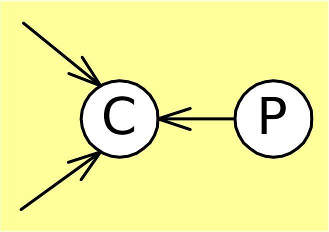
$$\frac{\partial f}{\partial x} = f_x = 3Ax^2 + 6Bxy + 3Cy^2 + 6Exw + 6Fyw + 3Hw^2$$

$$f_{xx} = 6Ax + 6By + 6Ew$$

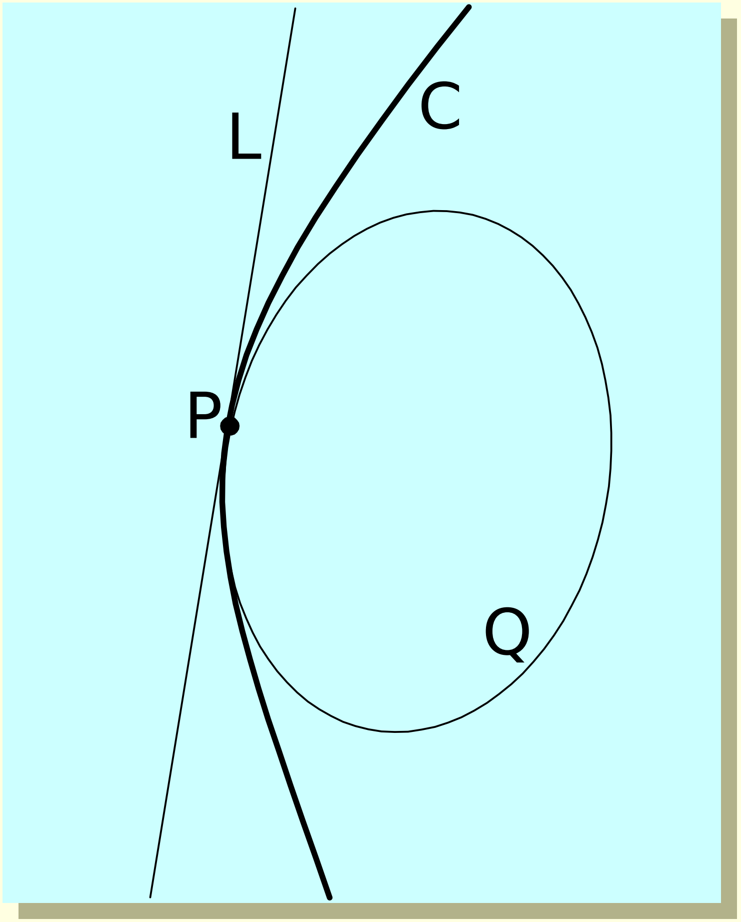
$$f_{xy} = 6Bx + 6Cy + 6Fw$$

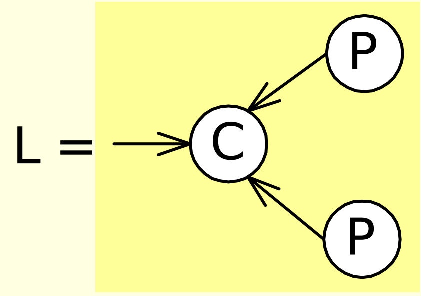
$$f_{xw} = 6Ex + 6Fy + 6Hw^2$$

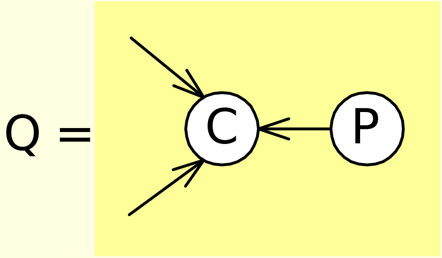
$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xw} \\ f_{xy} & f_{yy} & f_{yw} \\ f_{xw} & f_{yw} & f_{ww} \end{bmatrix} = 6$$



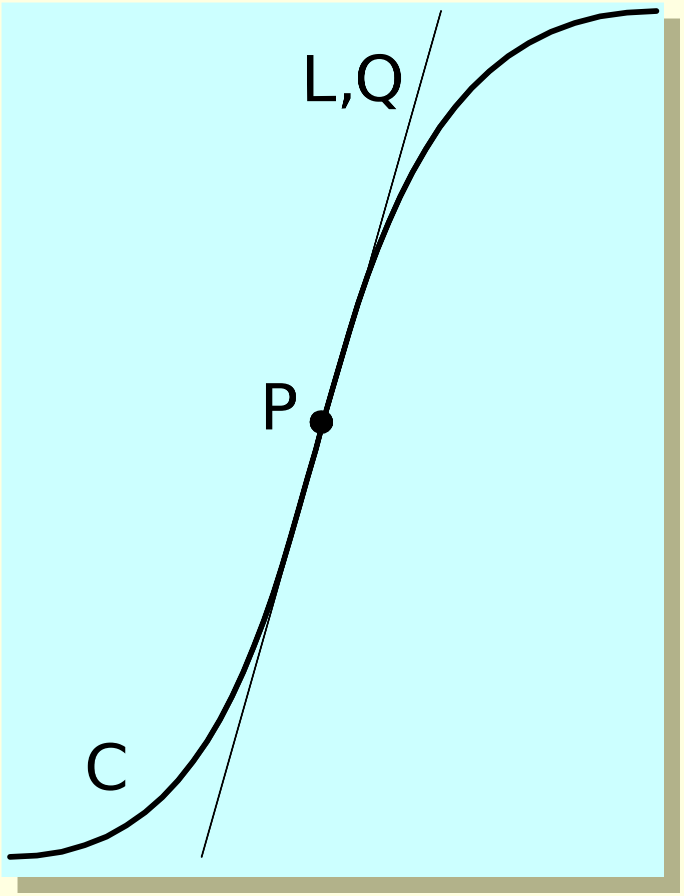
Typical Points

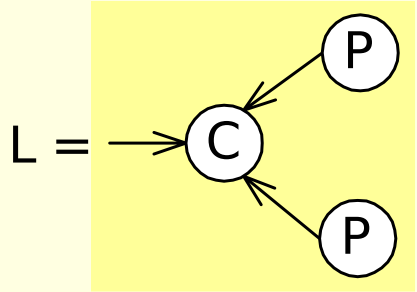


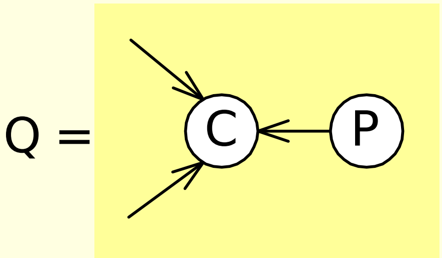




Inflection Points



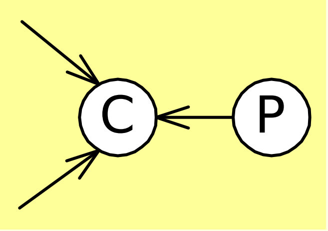




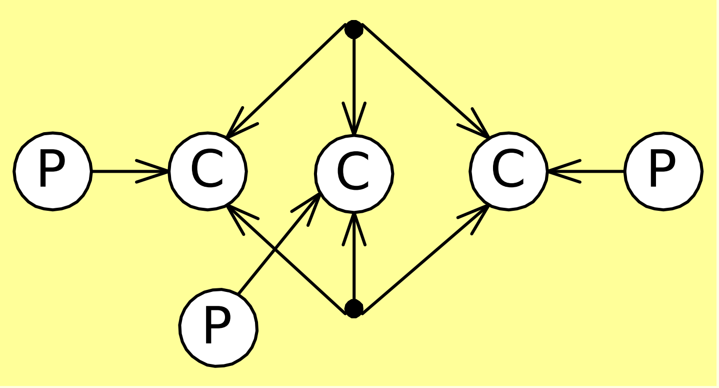
$$\det \mathbf{Q} = 0$$

Hessian

$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xw} \\ f_{xy} & f_{yy} & f_{yw} \\ f_{xw} & f_{yw} & f_{ww} \end{bmatrix} = 6$$

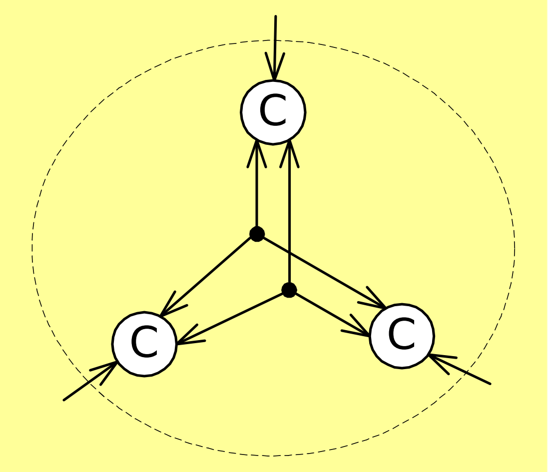


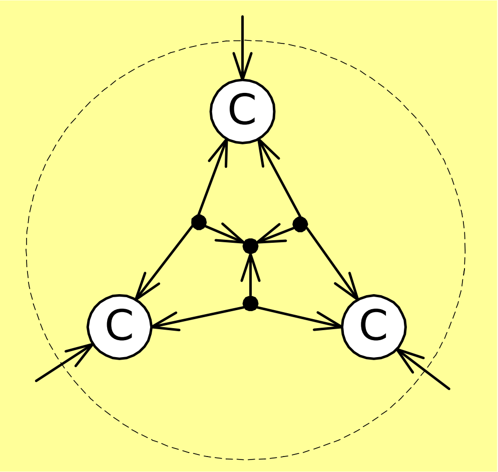
$$\mathbf{H}(x, y, w) = \det \begin{bmatrix} f_{xx} & f_{xy} & f_{xw} \\ f_{xy} & f_{yy} & f_{yw} \\ f_{xw} & f_{yw} & f_{ww} \end{bmatrix} =$$



$= 0$

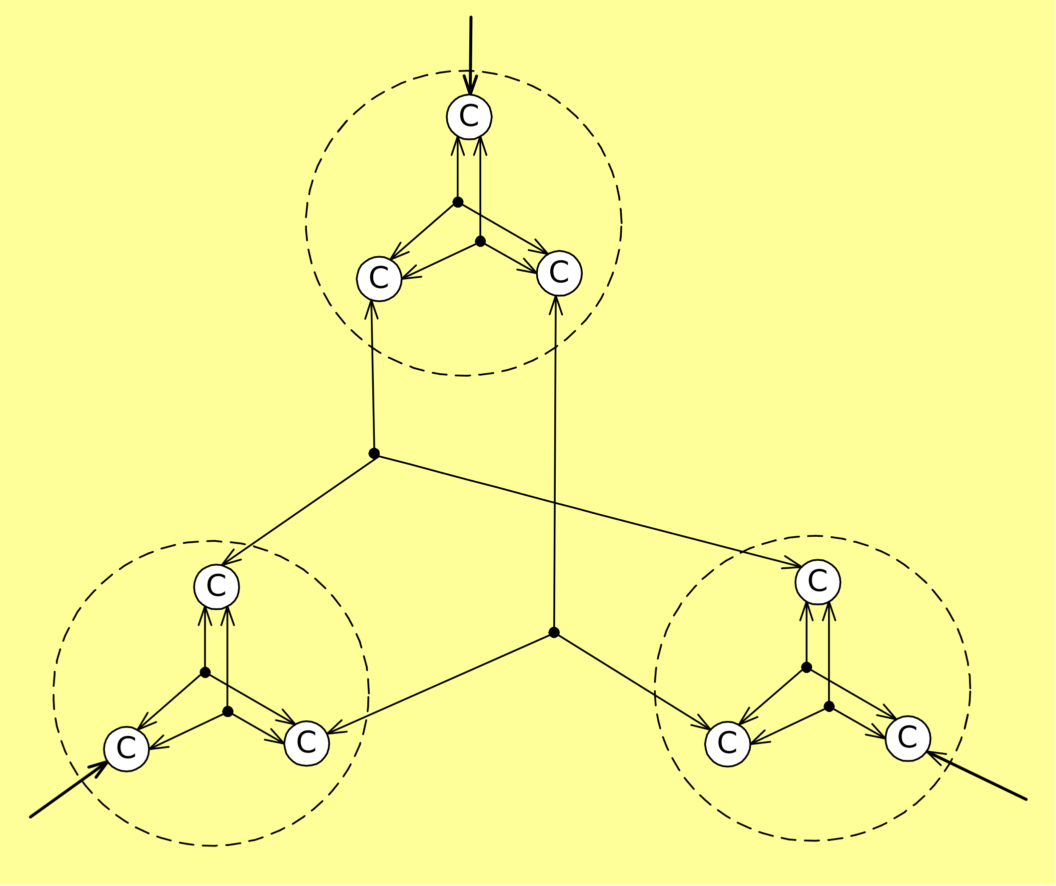
Hessian Diagram Forms





Note: Hessian transforms with original curve

Hessian of Hessian



Tensor Diagrams

Summary

Basic Points of Tensor Diagrams

- Fixes Representation Problems
  - Co/Contravariant
  - Higher Order with more prongs
- Manipulation Tools
  - Epsilon Delta identity
  - Substitution
- Representation of Invariant Quantities

Good Things

- Complicated polynomials have compact representation
- Aids visualization of algebraic structure
- Factoring is easier (local control)
- Suggests invariant quantities

Bad Things

- Combinatorial explosion for high orders and high dimensionality
- Resultants and Discriminants not as pretty as I would like

Tools for experimentation

- Diagram drawing program that can drag connected networks
- Symbolic algebra program that specializes in epsilons

Work to do

- Relate invariant diagrams to geometry
  (Geom to Dgm, Dgm to Geom)
  - Raw diagram fragments
  - Cross ratio generalizations
  - Not enough diagrams to cover all geometric cases
- Push to higher orders/dimensions